Probabilistic analysis of the Grassmann condition number

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Abstract

We perform an average analysis of the Grassmann condition number $\mathscr{C}(A)$ for the homogeneous convex feasibility problem $\exists x \in C \setminus 0 : Ax = 0$, where $C \subset \mathbb{R}^n$ may be any regular cone. This in particular includes the cases of linear programming, second-order programming, and semidefinite programming. We thus give the first average analysis of convex programming, which is not restricted to linear programming. The Grassmann condition number is a geometric version of Renegar's condition number, which we have introduced recently in [arXiv:1105.4049v1]. In this work we use techniques from spherical convex geometry and differential geometry. Among other things, we will show that if the entries of $A \in \mathbb{R}^{m\times n}$ are chosen i.i.d. standard normal, then for any regular cone C we have $\mathbb{E}[\ln \mathscr{C}(A)] < 1.5\ln(n) + 1.5$.

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1 Introduction

Convex programming is an efficient tool in modern applied mathematics. In fact, a commonly accepted technique in current scientific computing is to "convexify"supposedly hard problems, solve the relaxed convex problem, and then hope that the result is close to a solution of the original problem. To quote from [9, §1.3.2]: "With only a bit of exaggeration, we can say that, if you formulate a practical problem as a convex optimization problem, then you have solved the original problem."

But what is the complexity of convex programming? To specify this question further, we ask for the number of arithmetic operations, or the number of iterations of an interior-point method. Steve Smale suggested in [49] to use the concepts of condition numbers and probabilistic analysis in a 2-part scheme for the analysis of numerical algorithms: 1. Establish a bound for the running time, which is polynomial in the size of the input and (the logarithm

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of) a certain condition number of the input. 2. Analyze the condition number of a random input in form of tail estimates.

The first step of this scheme, i.e., the analysis of the role of condition numbers in convex programming, was initialized by Jim Renegar in [42, 43, 44], and is an active area of research, cf. [55, 56, 23, 26, 25, 39, 41, 17, 18, 21, 40, 24, 54, 6]. In these references the role of condition numbers is analyzed for linear and nonlinear convex programming, for exact arithmetic and for finite-precision arithmetic, for ellipsoid and for interior-point methods, etc.

Yet, the second step of Smale's scheme, i.e., the probabilistic analysis of the condition number, was until now severly restricted to the linear programming case. See the survey article [10] and the references given therein for more details on probabilistic analyses of condition numbers for linear programming.

We will give in this paper the first average analysis of a condition number for the general homogeneous convex feasibility problem. This includes the special cases of linear programming, second-order programming, and notably also the semidefinite programming case. More precisely, we consider the following problem:

Let $C \subset \mathbb{R}^n$ be a *regular cone*, i.e., C is a closed convex cone with nonempty interior that does not contain a nontrivial linear subspace. The *dual cone*[1] of C is defined as $\breve{C} := \{z \in \mathbb{R}^n \mid \forall x \in C : z^T x \leq 0\}$. We call C *self-dual* if $\breve{C} = -C$. The *homogeneous convex feasibility problem* is to decide for a given matrix $A \in \mathbb{R}^{m\times n}$, $1 \leq m < n$, the alternative[2]

$$\exists x \in \mathbb{R}^n \setminus 0 \ \text{ s.t.} \quad Ax = 0\,,\ x \in \breve{C}\,, \tag{P}$$

$$\exists y \in \mathbb{R}^m \setminus 0 \ \text{ s.t.} \quad A^T y \in C\,. \tag{D}$$

This problem reduces to the linear feasibility problem if $C = \mathbb{R}^n_+$, it reduces to the second-order feasibility problem if $C = \mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r}$, where $\mathcal{L}^n := \{x \in \mathbb{R}^n \mid x_n \geq x_1^2 + \cdots + \ldots, x_{n-1}^2\}$ denotes an n-dimensional Lorentz cone, and it reduces to the semidefinite feasibility problem if $C = \mathrm{Sym}^k_+ = \{M \in \mathrm{Sym}^k \mid M \succeq 0\}$ is the cone of positive semidefinite matrices, where $\mathrm{Sym}^k = \{M \in \mathbb{R}^{k\times k} \mid M^T = M\}$.

The condition number for which we will provide an average analysis, is the *Grassmann condition number*, that we have introduced in [3, 4], cf. also [6]. In the following paragraph we will recall the necessary definitions so that we can state the main theorem of this paper. See [4] for a more extensive description of the Grassmann condition and its relation to Renegar's condition number. We denote by $\mathrm{Gr}_{n,m}$ the set of m-dimensional linear subspaces in $\mathbb{R}^n$. It is a well-known fact that $\mathrm{Gr}_{n,m}$ is a smooth Riemannian manifold, called *Grassmann manifold*.

Definition 1.1. Let $C \subset \mathbb{R}^n$ be a regular cone, and let $1 \leq m \leq n-1$. We define the sets of *dual feasible* and *primal feasible* subspaces via

$$\mathcal{D}_m(C) := \{W \in \mathrm{Gr}_{n,m} \mid W \cap C \neq 0\}\,, \qquad \mathcal{P}_m(C) := \{W \in \mathrm{Gr}_{n,m} \mid W^\perp \cap \breve{C} \neq 0\}\,.$$

Furthermore, we define the set of *ill-posed* subspaces via

$$\Sigma_m(C) := \mathcal{D}_m(C) \cap \mathcal{P}_m(C)\,.$$

[1] Some authors call $\breve{C}$ the *polar cone*, and define the dual cone as $-\breve{C}$.

[2] In fact, (P) and (D) are only weak alternatives, as it may happen that both (P) and (D) are satisfiable. But the Lebesgue measure of the set of these ill-posed inputs in $\mathbb{R}^{m\times n}$ is zero.

It is known (cf. [4]) that $\mathcal{D}_m(C)$ and $\mathcal{P}_m(C)$ are compact subsets of $\mathrm{Gr}_{n,m}$, and $\mathcal{D}_m(C)\cup\mathcal{P}_m(C)=\mathrm{Gr}_{n,m}$. Moreover the boundaries of $\mathcal{D}_m(C)$ and $\mathcal{P}_m(C)$ coincide with $\Sigma_m(C)$. Furthermore,

$$\Sigma_m(C)=\{W\in\mathrm{Gr}_{n,m}\mid W\cap C\neq 0 \text{ and } W\cap\mathrm{int}(C)=\emptyset\}\ .$$

In other words, the set of ill-posed subspaces consists of those subspaces, which touch the cone C at the boundary. The following definition is a restatement of [4, Def. 1.2]. We denote by d_g the *geodesic distance* in $\mathrm{Gr}_{n,m}$ (see [4] for more details on this).

Definition 1.2. The *Grassmann condition number* is defined via

$$\mathscr{C}_C\colon\ \mathrm{Gr}_{n,m}\to[1,\infty]\ ,\quad \mathscr{C}_C(W):=\frac{1}{\sin d_g(W,\Sigma_m(C))}\ ,$$

where $d_g(W,\Sigma_m(C)):=\inf\{d_g(W,W')\mid W'\in\Sigma_m(C)\}$.

To simplify the notation we write $\mathscr{C}(W):=\mathscr{C}_C(W)$. Furthermore, for $A\in\mathbb{R}^{m\times n}$ with $\mathrm{rk}(A)=m$ we write $\mathscr{C}(A):=\mathscr{C}(\mathrm{im}\,A^T)$. Before we state the main result of this paper, we recall from [4] the following fundamental relation between the Grassmann condition number $\mathscr{C}(A)$ and Renegar's condition number $\mathcal{R}(A)$

$$\mathscr{C}(A)\ \le\ \mathcal{R}(A)\ \le\ \kappa(A)\cdot\mathscr{C}(A)\ , \tag{1}$$

where $\kappa(A)$ denotes the usual matrix condition number, i.e., the ratio between the largest and the smallest singular value of A. The Grassmann condition number can be interpreted as a coordinate-free version of Renegar's condition number, and the dependence of Renegar's condition number on the representation of the subspace is conveniently described by the matrix condition number $\kappa(A)$. See [4] for more details.

The inequalities (1) show that probabilistic analyses of $\mathscr{C}$ easily imply corresponding results about $\mathcal{R}$, as the probabilistic behavior of the usual matrix condition number is a well-studied subject, cf. [20, 16, 13]. For example, in [16] it was shown that for Gaussian matrices $A\in\mathbb{R}^{m\times n}$ one has

$$\mathbb{E}\,[\ln\kappa(A)]\ \ <\ \ \ln\Big(\frac{n}{1+|n-m|}\Big)+2.3\ . \tag{2}$$

The following two theorems are the main results of this paper.

Theorem 1.3. *Let $C\subset\mathbb{R}^n$ be a regular cone with $n\ge 3$. If $A\in\mathbb{R}^{m\times n}$ is a Gaussian random matrix, then we have*

$$\mathrm{Prob}[\mathscr{C}(A)>t]\ \ <\ \ 6\cdot\sqrt{m(n-m)}\cdot\frac{1}{t}\ ,\quad \text{if } t>n^{\frac{3}{2}}\ , \tag{3}$$

$$\mathbb{E}\,[\ln\mathscr{C}(A)]\ \ <\ \ 1.5\cdot\ln(n)+1.5\ . \tag{4}$$

In fact, these bounds hold for any probability distribution on $\mathbb{R}^{m\times n}$, which induces the uniform distributions on the Grassmann manifolds $\mathrm{Gr}_{n,n-m}$ and $\mathrm{Gr}_{n,m}$ via the maps $A\mapsto\ker A$ and $A\mapsto\mathrm{im}\,A^T$. We have reduced our formulation to the case of Gaussian random matrices for simplicity.

Assuming certain conjectures on the growth of the intrinsic volumes of special cones C (see Section 2.3) we can improve the bounds as follows.

Theorem 1.4. *Let $C \subset \mathbb{R}^n$ be one of the following cones: $\mathbb{R}^n_+$, $\mathcal{L}^n$, $\mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r}$, Sym^k_+. If $A \in \mathbb{R}^{m\times n}$, with $m \geq 8$, is a Gaussian random matrix, then we have*

$$\begin{aligned} \mathrm{Prob}[\mathscr{C}(A) > t] &< 20 \cdot v(C) \cdot \sqrt{m} \cdot \frac{1}{t} \;, \quad \text{if } t > m \;, & (5)\\ \mathbb{E}\left[\ln \mathscr{C}(A)\right] &< \ln(m) + \max\{\ln(v(C)), 0\} + 3 \;, & (6) \end{aligned}$$

where the excess over the Lorentz cone $v(C)$ (cf. (28) for the definition) is bounded as shown in the following table

C	$\mathbb{R}^n_+$	$\mathcal{L}^n$	$\mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r}$ *(assuming Conjecture 2.17)*	Sym^k_+ *(assuming Conjecture 2.18)*
$v(C) \leq$	$\sqrt{2}$	1	2^{r-1}	2

.

To illustrate the significance of Theorem 1.3, we will use a result from [54] to state a complexity result about general convex programming.

Corollary 1.5. *Let C be a self-scaled cone with a self-scaled barrier function. Then there exists an interior point algorithm, that solves the general homogeneous convex feasibility problem for Gaussian random inputs in expected $O(\sqrt{\nu_C} \cdot (\ln(\nu_C) + \ln(n)))$ number of interior-point iterations. Here, ν_C denotes the complexity parameter of the barrier function for the reference cone C. For linear programming, second-order programming, and semidefinite programming, we have an expected number of interior-point iterations of $O(\sqrt{n} \cdot \ln(n))$.*

Proof. In [54] the authors describe an interior-point algorithm that solves the general homogeneous convex feasibility problem, for C a self-scaled cone with a self-scaled barrier function, in $O(\sqrt{\nu_C} \cdot \ln(\nu_C \cdot \mathcal{R}(A)))$ interior-point iterations. From (1) we have $\ln(\mathcal{R}(A)) \leq \ln(\kappa(A)) + \ln(\mathscr{C}(A))$. Using the estimates (2) and (4) we thus get $\mathbb{E}[\ln(\mathcal{R}(A))] = O(\ln(n))$. As for the complexity parameter ν_C, the typical barrier functions for (LP), (SOCP), (SDP), yield

$$C = \mathbb{R}^n_+ \;:\; \nu_C = n \;, \qquad C = \mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r} \;:\; \nu_C = 2\,r \;, \qquad C = \mathrm{Sym}^k_+ \;:\; \nu_C = k \;.$$

In particular, for these cases we have $\nu_C \leq n$. □

Additionally, we mention that in [54] it is also shown that the condition number of the system of equations, that is solved in each interior-point iteration, is bounded by a factor of $\mathcal{R}(A)^2$. One can thus also use our results to get information about the expected cost of each iteration in the above-mentioned algorithm. We leave this to the interested reader.

This paper overall consists of two parts. The first part, which consists of the Sections 2–3, is elementary and easily accessible. In Section 2.1 we review some facts about spherical intrinsic volumes and in Section 2.3 we explain how these may be estimated. In Section 3 we use the fact that $\mathrm{Gr}_{n,m}$ is a Riemannian manifold, which allows to define the *tube* $\mathcal{T}(\Sigma_m, \alpha)$ of radius α around Σ_m with respect to the geodesic distance d_g, cf. (33). We will state in Theorem 3.1 an estimate of the volume of $\mathcal{T}(\Sigma_m, \alpha)$, and we shall derive proofs for Theorem 1.3 and Theorem 1.4 from this estimate.

The second part of the paper is devoted to the proof of Theorem 3.1. In Section 4 we will define a certain generalization $\mathrm{ch}_Y(\varphi, t)$ of the characteristic polynomial of a self-adjoint linear

operator $\varphi\colon V \to V$ of a euclidean vector space V, which depends on a linear subspace $Y \subseteq V$. We call $\mathrm{ch}_Y(\varphi, t)$ the *twisted characteristic polynomial* and we will compute its expectation when the subspace Y (of a fixed dimension) is chosen uniformly at random. Section 4 only requires some elementary linear algebra, and although it seems to lie outside the general framework of this paper, it is a central step towards the proof of Theorem 3.1. In Section 5 we will provide some preliminaries from Riemannian geometry, and finally, in Section 6 we will complete the proof of Theorem 3.1.

From a high level point of view, one may interpret Theorem 3.1 as an extension of *Weyl's (spherical) tube formula*, cf. [57], to the Grassmann manifold setting. Although such extensions in the sense of Theorem 3.1 were known before, cp. [28, 29], these have the serious drawback that they only hold for radii α below a certain threshold α_0, which depends on the cone C. In fact, for the interesting cones used in convex programming, this threshold is $\alpha_0 = 0$, so that the previously existing results are useless for applications in convex programming, cf. Remarks 2.1 and 3.2. Theorem 3.1 does not suffer from this restriction and may thus be interpreted as a less precise, but more robust version of Weyl's tube formula in the Grassmann manifold setting, like it was achieved for the spherical setting in [14].

A key idea in the proof of Theorem 3.1 is to consider cones with *smooth* boundaries. More precisely, we will assume that $M := \partial C \cap S^{n-1}$ is a smooth hypersurface of S^{n-1} with strictly positive curvature. Then it turns out that the corresponding set $\Sigma_m(C)$ of ill-posed inputs is an embedding of the Grassmann bundle $\mathrm{Gr}(M, m-1)$ in the Grassmann manifold $\mathrm{Gr}_{n,m}$. Exploiting the induced bundle structure on Σ_m is the crucial idea for the proof of Theorem 3.1.

2 Spherical convex geometry

It is a simple, yet essential observation that the analysis of the question whether a subspace hits a convex cone nontrivially, may be transferred from euclidean space to spherical space by intersecting both the subspace and the cone with the unit sphere. The above question then translates into the question of whether a subsphere of the unit sphere intersects a spherical convex set, or not. The analysis of the homogeneous convex feasibility problem thus naturally finds its place in the domain of spherical convex geometry.

While euclidean convex geometry is a classical and extensively studied subject, the situation for spherical convex geometry is quite different. The theory here is much less established, while a number of results, which are difficult to find in the literature, seem to be folklore among the experts in convex geometry. We find it thus appropriate to recall in this section a number of elementary facts and definitions to provide a preferably self-contained presentation. In our presentation we rely on the theses [28] (cf. also [29]) and [3], and the survey article [27]. We will put a special emphasis on *intrinsic volumes*.

The first subsection is mainly devoted to the elementary definitions and the corresponding notations. In the second subsection we will give the definition of spherical intrinsic volumes, and we will present formulas for the intrinsic volumes of the cones $\mathbb{R}^n_+$, $\mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_r}$, Sym^k_+. In the third subsection we will describe a general way to estimate the intrinsic volumes (of self-dual cones). These estimates will be used in the proof of Theorem 1.4, which we will give in Section 3.

2.1 Some general facts

We denote the spherical distance function by $d\colon S^{n-1}\times S^{n-1}\to\mathbb{R}$, $d(p,q)=\arccos(\langle p,q\rangle)$. For a subset $K\subseteq S^{n-1}$ of the unit sphere we denote the cone generated by this set by $\operatorname{cone}(K):=\{\lambda\cdot p\mid\lambda\geq 0\,,\ p\in K\}$, and we additionally set $\operatorname{cone}(\emptyset):=\{0\}$. A subset $K\subseteq S^{n-1}$ is called *(spherical) convex* iff for all $p,q\in K$ with $q\neq\pm p$ the (unique geodesic) arc between p and q is contained in K. This is equivalent to the condition that the set $\operatorname{cone}(K)$ is a convex cone. We denote the family of closed spherical convex sets by $\mathcal{K}(S^{n-1})$. Note that the empty set as well as the whole sphere S^{n-1} both lie in $\mathcal{K}(S^{n-1})$.

For $M\subseteq S^{n-1}$ closed and $\alpha\geq 0$ we denote the *tube* of radius α around M by $\mathcal{T}(M,\alpha):=\{p\in S^{n-1}\mid\exists q\in M: d(p,q)\leq\alpha\}$. Using the notion of tubes we may define the Hausdorff distance d_{H} on $\mathcal{K}(S^{n-1})$ in the following way

$$d_{\mathrm{H}}(K_1,K_2):=\max\big\{\min\{\alpha\geq 0\mid K_2\subseteq\mathcal{T}(K_1,\alpha)\}\,,\ \min\{\beta\geq 0\mid K_1\subseteq\mathcal{T}(K_2,\beta)\}\big\}\ ,$$

for $K_1,K_2\in\mathcal{K}(S^{n-1})$. It is well-known (cf. for example [38, §1.2]) that the map d_{H} is a metric on $\mathcal{K}(S^{n-1})$, which turns $\mathcal{K}(S^{n-1})$ into a compact metric space.

The duality map which sends a closed convex cone C onto its dual cone $\breve{C}$ naturally defines an involution on $\mathcal{K}(S^{n-1})$, which we also denote by $\breve{.}\colon\mathcal{K}(S^{n-1})\to\mathcal{K}(S^{n-1})$, and which is given by $\breve{K}:=\breve{C}\cap S^{n-1}=\{p\in S^{n-1}\mid d(p,K)\geq\frac{\pi}{2}\}$, where $C:=\operatorname{cone}(K)$. It can be shown (cf. for example [28, Hilfssatz 2.2] or [3, Prop. 3.2.4]) that for $K_1,K_2\in\mathcal{K}(S^{n-1})$ with $d_{\mathrm{H}}(K_1,K_2)<\frac{\pi}{2}$ we have $d_{\mathrm{H}}(\breve{K}_1,\breve{K}_2)=d_{\mathrm{H}}(K_1,K_2)$. In particular, the duality map is a local isometry.

Remark 2.1. One important difference between euclidean and spherical convex geometry are the convexity properties of tubes. In the euclidean case the tubes around a convex set are again convex. In the spherical case it is rarely true that the tube around a convex set is again convex, unless the tube is the whole sphere. For example, it is elementary to show (cf. [3, §3.1.2]) that if $K\in\mathcal{K}(S^{n-1})$ contains two antipodal points, or if the boundary of K contains a geodesic segment, then $\mathcal{T}(K,\alpha)\in\mathcal{K}(S^{n-1})$ for some $\alpha>0$ implies $\mathcal{T}(K,\alpha)=S^{n-1}$. It follows that tubes around (proper) subspheres are either the whole sphere or not convex. The same statement holds for spherical convex sets K, whose corresponding cone $C=\operatorname{cone}(K)$ has a nontrivial face, i.e., there exists a supporting hyperplane $H\subset\mathbb{R}^n$, $H\cap\operatorname{int}(C)=\emptyset$, such that the (linear hull of the) face $F:=H\cap C$ has dimension $\dim\operatorname{lin}(F)\geq 2$. Hence, among our standard ensemble of cones $\mathbb{R}^n_+$, $\mathcal{L}^{n_1}\times\ldots\times\mathcal{L}^{n_r}$, Sym^k_+, only the cones $\mathcal{L}^n$, $n\in\mathbb{N}$, have convex tubes different from the whole sphere.

Another difference between euclidean and spherical convex geometry is that while the set of all convex bodies, i.e., the set of all compact euclidean convex sets, is connected, this is not true for $\mathcal{K}(S^{n-1})$ as we will see next. We use the notation $\mathcal{S}^{-1}(S^{n-1}):=\{\emptyset\}$, and

$$\begin{aligned}\mathcal{S}^k(S^{n-1})&:=\{S\subset S^{n-1}\mid S\text{ is a }k\text{-dim. subsphere}\}\ ,\quad k=0,1,\ldots,n-1\ ,\\ \mathcal{K}^c(S^{n-1})&:=\{K\in\mathcal{K}(S^{n-1})\mid K\neq\emptyset\text{ and }K\text{ is not a subsphere}\}\ .\end{aligned}$$

Note that there is a canonical bijection between $\mathcal{S}^k(S^{n-1})$ and $\mathrm{Gr}_{n,k+1}$ given by $\mathrm{Gr}_{n,k+1}\to\mathcal{S}^k(S^{n-1})$, $W\mapsto W\cap S^{n-1}$, and its inverse $\mathcal{S}^k(S^{n-1})\to\mathrm{Gr}_{n,k+1}$, $S\mapsto\operatorname{lin}(S)$. Note also that $\breve{S}=S^\perp:=W^\perp\cap S^{n-1}$ for S a subsphere of S^{n-1}, with $W:=\operatorname{lin}(S)$.

As the non-subspheres are central objects for our study, we call elements in $\mathcal{K}^c(S^{n-1})$ *caps.* This naming is different from other works, where "cap" may denote a spherical ball (cf. for

example [11]). We will denote spherical balls by the term *circular caps*. It is easily checked that an element $K \in \mathcal{K}(S^{n-1})$ is a cap iff $\exists p \in K : -p \notin K$.

The set of spherical convex sets $\mathcal{K}(S^{n-1})$ decomposes into caps and subspheres, i.e., the decomposition of $\mathcal{K}(S^{n-1})$ into its connected components is given by

$$\mathcal{K}(S^{n-1}) = \mathcal{K}^c(S^{n-1}) \ \dot\cup \ \dot{\bigcup_{k=-1}^{n-1}} \mathcal{S}^k(S^{n-1}) \,. \tag{7}$$

The connectedness of the components in this decomposition is checked easily. The fact that elements from separate components cannot be connected via a path in $\mathcal{K}(S^{n-1})$ is seen in the following way. If W_1, W_2 are subspaces of $\mathbb{R}^n$ of different dimensions, then counting dimensions shows that $W_1 \cap W_2^\perp \neq 0$ or $W_1^\perp \cap W_2 \neq 0$. This implies that the corresponding subspheres $S_i := W_i \cap S^{n-1}$, $i = 1, 2$, have Hausdorff distance $d_{\mathrm{H}}(S_1, S_2) \geq \frac{\pi}{2}$. In fact, as $\mathcal{T}(S, \frac{\pi}{2}) = S^{n-1}$ for any (nonempty) subsphere S of S^{n-1}, we have $d_{\mathrm{H}}(S_1, S_2) = \frac{\pi}{2}$ for subspheres of different dimensions. For a cap $K \in \mathcal{K}^c(S^{n-1})$ and a subsphere S we have the well-known theorem of alternatives, also known as *Farkas-Lemma*, which says that $K \cap S \neq \emptyset$ or $\breve{K} \cap S^\perp \neq \emptyset$. This implies $d_{\mathrm{H}}(K, S) \geq \frac{\pi}{2}$. The decomposition in (7) is thus the decomposition $\mathcal{K}(S^{n-1})$ in connected components.

The set of caps $\mathcal{K}^c(S^{n-1})$ should be seen as the essential part of $\mathcal{K}(S^{n-1})$ containing a variety of sets with diverse properties. We need to specify subfamilies of $\mathcal{K}(S^{n-1})$ with which we can work in a unified way. We begin with the set of *polyhedral convex sets*: A spherical convex set $K \in \mathcal{K}(S^{n-1})$ is called *polyhedral* if $\mathrm{cone}(K)$ is the intersection of finitely many half-spaces:

$$\mathcal{K}^p(S^{n-1}) := \{K \in \mathcal{K}(S^{n-1}) \mid \mathrm{cone}(K) = H_1 \cap \ldots \cap H_k \,,\ H_i \text{ is a closed half-space in } \mathbb{R}^n\}.$$

Equivalently (cf. for example [45, Sec. 19]), a set $K \in \mathcal{K}(S^{n-1})$ is polyhedral iff it is the (spherical) convex hull of finitely many points in S^{n-1}. Our standard example for a polyhedral convex set is the intersection $\mathbb{R}^n_+ \cap S^{n-1}$ of the positive orthant with the unit sphere. By definition, the set of polyhedral convex sets contains the subspheres of S^{n-1}. Furthermore, the set of polyhedral convex sets lies dense in $\mathcal{K}(S^{n-1})$ with respect to the Hausdorff metric. This is seen by an easy adaption (cf. [28, Hilfssatz 2.5] or [3, Prop. 3.3.4]) of the proof for the euclidean statement (cf. [47, §2.4]).

Another important subfamily of spherical convex sets is given by the set of *regular caps*: A spherical convex set $K \in \mathcal{K}(S^{n-1})$ is called *regular* if both K and $\breve{K}$ have nonempty interior:

$$\mathcal{K}^r(S^{n-1}) := \{K \in \mathcal{K}(S^{n-1}) \mid \mathrm{int}(K) \neq \emptyset \text{ and } \mathrm{int}(\breve{K}) \neq \emptyset\} \,.$$

Note that subspheres are not regular, i.e., the set of regular convex sets is a subset of the set $\mathcal{K}^c(S^{n-1})$ of caps, and we may thus speak of *regular caps*. In the set $\mathcal{K}^r(S^{n-1})$ of regular caps we define for $n \geq 3$ the subclass of *smooth caps* via

$$\mathcal{K}^{\mathrm{sm}}(S^{n-1}) := \left\{ K \in \mathcal{K}^r(S^{n-1}) \;\middle|\; \begin{array}{c} \partial K \text{ is a smooth hypersurface in } S^{n-1} \\ \text{with nowhere vanishing Gaussian curvature} \end{array} \right\} .$$

Here, the Gaussian curvature is the determinant of the *Weingarten map* (see for example [53] for an elementary introduction of these notions; see [19, Ch. 6] for a more thorough treatment).

For hypersurfaces M of S^{n-1} with a unit normal vector field $\nu\colon M \to S^{n-1}$ the Weingarten map at $p \in M$ is given by

$$\mathcal{W}_p\colon T_pM \to T_pM\ ,\ \mathcal{W}_p(\zeta) = -D_p\nu(\zeta)\ , \tag{8}$$

where $D_p\nu$ denotes the derivative of ν at p (for hypersurfaces of $\mathbb{R}^n$ this is elementary (cf. for example [53, Ch. 9]); for hypersurfaces of S^{n-1} this is also true (cf. [3, Sec. 4.1.1])). It can be shown that the Weingarten map is self-adjoint, and we will denote its eigenvalues by $\kappa_1(p),\ldots,\kappa_{n-2}(p)$. These are called the *principal curvatures of M at p*. Furthermore, we will denote by $\sigma_k(p)$, $0 \le k \le n-2$, the kth elementary symmetric function in the eigenvalues of $\mathcal{W}_p$. When M is the smooth boundary of a cap $M = \partial K$ with $\operatorname{int}(K) \neq \emptyset$, then we will always assume that $\nu(p)$ denotes the normal direction pointing inwards the cap K. This has the effect that the Weingarten map is positive definite, i.e., M has everywhere positive curvature (cf. [47, §2.5] for the corresponding euclidean statement).

Example 2.2. Let $M = \partial K \subset S^{n-1}$ be the boundary of a circular cap $K := B(z,\beta) = \{p \in S^{n-1} \mid d(z,p) \le \beta\}$ of radius $\beta \in (0,\pi)$. Then the principal curvatures $\kappa_1(p),\ldots,\kappa_{n-2}(p)$ at $p \in M$ are given by $\kappa_1(p) = \ldots = \kappa_{n-2}(p) = \cot(\beta)$. In particular, we have $\sigma_i(p) = \binom{n-2}{i}\cdot\cot(\beta)^i$.

For the euclidean case, Minkowski (cf. [7, §6]) has shown that the set of smooth convex bodies lies dense in the set of convex bodies. Using this result, it is elementary to show (see for example [3, Prop. 4.1.10]) that the set of smooth caps $\mathcal{K}^{\mathrm{sm}}(S^{n-1})$ lies dense in the set of all caps $\mathcal{K}^c(S^{n-1})$.

We finish this section with an important property of smooth caps, that we will need in the proof of the main result in Section 6.

Proposition 2.3. *Let $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$ and $C := \operatorname{cone}(K)$. If $W \in \Sigma_m(C)$, then $W \cap K = \{p\}$ for some $p \in \partial K$.*

Proof. As $W \in \Sigma_m(C)$, we have $\Sigma_m \cap C = \Sigma_m \cap \partial C \neq \{0\}$. It follows that there exists $p \in W \cap K$. To prove that p is the only element in $W \cap K$, assume that there exists $q \in W \cap K$, $p \neq q$. As K is a regular cap, we have $p \neq -q$, so that there exists a unique geodesic arc between p and q. By convexity of W and K, this arc lies in $W \cap K$, and thus in M, the boundary of K. But this implies that along this arc M has zero Gaussian curvature, which contradicts the assumption $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$. □

2.2 Intrinsic volumes

Before we give the definition of spherical intrinsic volumes we introduce the following notation for the volume of the unit sphere and for the volume of tubes around subspheres

$$\begin{aligned} \mathcal{O}_{n-1} &:= \operatorname{vol}_{n-1}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \\ \mathcal{O}_{n-1,k}(\alpha) &:= \operatorname{vol}_{n-1}\mathcal{T}(S,\alpha)\ , \end{aligned} \tag{9}$$

where $S \in \mathcal{S}^k(S^{n-1})$, and $0 \le \alpha \le \frac{\pi}{2}$. An elementary computation shows (see for example [12, Lem. 20.5] or [3, Prop. 4.1.18]) that the volume of the α-tube $\mathcal{T}(S,\alpha)$ is given by

$$\mathcal{O}_{n-1,k}(\alpha) = \mathcal{O}_k \cdot \mathcal{O}_{n-2-k} \cdot \int_0^\alpha \cos(\rho)^k \cdot \sin(\rho)^{n-2-k}\, d\rho\ . \tag{10}$$

In particular, the volume of a circular cap $B(z,\beta)$ of radius $\beta \in [0,\pi]$ is given by

$$\mathrm{vol}_{n-1} B(z,\beta) = \mathcal{O}_{n-2} \cdot \int_0^\beta \sin(\rho)^{n-2}\, d\rho \,. \tag{11}$$

The following proposition may be used for the definition of the spherical intrinsic volumes (cf. [27]); it may be interpreted as a spherical version of the *Steiner polynomial* (cf. for example [47]).

Proposition 2.4. *For $K \in \mathcal{K}(S^{n-1}) \setminus \{\emptyset\}$ and $0 \le \alpha \le \frac{\pi}{2}$ the volume of the α-tube around K is given by*

$$\mathrm{vol}_{n-1} \mathcal{T}(K,\alpha) = \mathrm{vol}_{n-1}(K) + \sum_{j=0}^{n-2} V_j(K) \cdot \mathcal{O}_{n-1,j}(\alpha) \,, \tag{12}$$

for some uniquely determined continuous functions $V_j \colon \mathcal{K}(S^{n-1}) \to \mathbb{R}$, $0 \le j \le n-2$.

Proof. See (one of) the references given in [27]: [31, 2, 46, 36, 28]. □

Definition 2.5. For $-1 \le j \le n-1$ the jth *(spherical) intrinsic volume* is the function $V_j \colon \mathcal{K}(S^{n-1}) \to \mathbb{R}$, defined as follows. For $0 \le j \le n-2$ it is defined via

$$V_j(K) := \begin{cases} \text{the quantity defined in (12)} & \text{if } K \neq \emptyset \,, \\ 0 & \text{if } K = \emptyset \,. \end{cases} \tag{13}$$

Furthermore, for $j \in \{-1, n-1\}$, the jth intrinsic volume of K is defined via

$$V_{n-1}(K) := \begin{cases} \frac{\mathrm{vol}_{n-1}(K)}{\mathcal{O}_{n-1}} & \text{if } K \neq \emptyset \\ 0 & \text{if } K = \emptyset \end{cases} \,, \qquad V_{-1}(K) := V_{n-1}(\breve{K}) \,. \tag{14}$$

The definition of $V_{-1}(K)$ in (14) may seem artificial at first sight, but in fact it is natural, as will become clear in (15), (16), (17) below. We will use the notation $V_j(C) := V_j(C \cap S^{n-1})$, if $C \subseteq \mathbb{R}^n$ is a closed convex cone, and refer to $V_j(C)$ as the jth intrinsic volume of the cone C. Furthermore, we use the notation $V(K)$ for the $(n+1)$-tuple of the intrinsic volumes, i.e. $V(K) := (V_{-1}(K), V_0(K), \ldots, V_{n-1}(K))$, and similarly for $V(C)$. As for the notation, it turns out that usually the formulas become nicer when using the shifted index $V_{j-1}(C)$ with $0 \le j \le n$. However, we refrain from reindexing the functionals V_j to simplify the notation, as the given definition is by now established (cf. [28, 27, 48, 3]), and a redefinition could lead to major confusion.

The following proposition provides a simple and illustrative description of the spherical intrinsic volumes in the special case of polyhedral cones. Additionally, a couple of essential properties of the intrinsic volumes are immediate from this description.

Proposition 2.6. *Let $C \subseteq \mathbb{R}^n$ be a polyhedral cone and let $\Pi_C \colon \mathbb{R}^n \to C$ denote the projection map onto C. Furthermore, let d_C denote the function*

$$d_C \colon \mathbb{R}^n \to \{0,1,2,\ldots,n\} \,, \quad x \mapsto \dim(\mathrm{face}(\Pi_C(x))) \,,$$

where face(x) *denotes the face of C of which x lies in its relative interior. Then the $(j-1)$th intrinsic volume of C, $0 \le j \le n$, is given by*

$$V_{j-1}(C) = \operatorname*{Prob}_{p \in S^{n-1}} [d_C(p) = j] = \operatorname*{Prob}_{x \in \mathcal{N}(0,I_n)} [d_C(x) = j] \,, \tag{15}$$

where $p \in S^{n-1}$ is drawn uniformly at random and $x \in \mathbb{R}^n$ is drawn at random according to the normal distribution $\mathcal{N}(0, I_n)$.

Proof. This was shown in [3, Prop. 4.4.6]. The statement is essentially a reinterpretation of a well-known formula for the spherical intrinsic volumes of polyhedral cones, cf. for example [27, Eq. (11)]. □

Remark 2.7. 1. From Proposition 2.6 it is immediate that the intrinsic volumes satisfy

$$\forall j = 0, \ldots, n : V_{j-1}(K) \geq 0 \;, \qquad \sum_{j=0}^{n} V_{j-1}(K) = 1 \tag{16}$$

for all $K \in \mathcal{K}(S^{n-1})$. The vector $V(K)$ may thus be interpreted as a probability distribution on the set $\{0, 1, \ldots, n\}$.

2. Another consequence of Proposition 2.6 is the following nice duality property of intrinsic volumes

$$V_{j-1}(\breve{C}) = V_{n-j-1}(C) \;, \tag{17}$$

where $0 \leq j \leq n$. This relation may be deduced (cf. [3, Prop. 4.4.10]) from the duality relation between the face lattices of C and of $\breve{C}$, respectively.

Remark 2.8. It is known that the *Euler characteristic* χ on $\mathcal{K}(S^{n-1})$ has the representation

$$\chi = 2 \cdot \sum_{\substack{j=0 \\ j \text{ odd}}}^{n} V_{j-1} \;.$$

(See [28, Sec. 4.3] or [48, Thm. 6.5.5] for proofs of this fact.) Since $\chi(K) = 1$ for all caps $K \in \mathcal{K}^c(S^{n-1})$, it follows that the intrinsic volumes of caps are bounded by $V_{j-1}(K) \leq \frac{1}{2}$.

Another simple consequence of Proposition 2.6 is a calculation rule for direct products of cones. The euclidean analog of this statement is well-known (cf. for example [34, Sec. 9.7]); the spherical statement below was, to our best knowledge, for the first time given in [3].

Corollary 2.9. *Let C_1, C_2 be closed convex cones. Then the intrinsic volumes of the direct product $C_1 \times C_2$ are given by*

$$V(C_1 \times C_2) = V(C_1) * V(C_2) \;, \tag{18}$$

where $*$ denotes the convolution operator, i.e., $V_{k-1}(C_1 \times C_2) = \sum_{i+j=k} V_{i-1}(C_1) \cdot V_{j-1}(C_2)$.

Proof (Sketch). We sketch the main ideas of the proof and refer for a more elaborated proof to [3, App. B.1]. By continuity of the intrinsic volumes, it suffices to consider the case where both C_1 and C_2 are polyhedral cones. The faces of $C_1 \times C_2$ are given by direct products of faces of C_1 and C_2. Furthermore, using the notation of Proposition 2.6, it is easy to see that $d_{C_1 \times C_2}(x_1, x_2) = d_{C_1}(x_1) + d_{C_2}(x_2)$. Hence, it follows from the characterization (15) that the intrinsic volumes of $C_1 \times C_2$ arise as the convolution of the intrinsic volumes of C_1 and C_2. □

Example 2.10. The calculation rule from Corollary 2.9 provides an easy way to compute the intrinsic volumes of the positive orthant: For $n=1$ we have $V(\mathbb{R}_+) = (\frac{1}{2},\frac{1}{2})$. As $\mathbb{R}_+^n = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$ we may apply (18) to get

$$V(\mathbb{R}_+^n) = (\tfrac{1}{2},\tfrac{1}{2}) * \ldots * (\tfrac{1}{2},\tfrac{1}{2}) = \left(\tbinom{n}{0}/2^n, \tbinom{n}{1}/2^n, \ldots, \tbinom{n}{n}/2^n\right) ,$$

i.e., we have $V_{j-1}(\mathbb{R}_+^n) = \binom{n}{j}/2^n$ for $0 \le j \le n$.

In the following proposition we give a well-known formula for the spherical intrinsic volumes of smooth caps, which goes back to H. Weyl [57]. (See [57], [14] or [3, Ch. 4] for proofs.)

Proposition 2.11. *Let $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$ and $1 \le j \le n-1$. Then the intrinsic volumes of K are given by*

$$V_{j-1}(K) \;=\; \frac{1}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{n-j-1}} \cdot \int_{p \in M} \sigma_{n-j-1}(p)\, dM \;, \tag{19}$$

where $M := \partial K$ denotes the boundary of K, and $\sigma_k(p)$ denotes the kth elementary symmetric function in the principal curvatures of M. □

Example 2.12. Let $K = B(z,\beta)$, $0 < \beta \le \pi/2$, be a circular cap. Then from (19) and from Example 2.2 we get for $1 \le j \le n-1$, with $M := \partial K$,

$$\begin{aligned} V_{j-1}(K) &= \frac{1}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{n-j-1}} \cdot \int_{p \in M} \binom{n-2}{j-1} \cdot \cot(\beta)^{n-j-1}\, dM \\ &= \frac{\mathcal{O}_{n-2}}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{n-j-1}} \cdot \binom{n-2}{j-1} \cdot \sin(\beta)^{j-1} \cdot \cos(\beta)^{n-j-1} \;, \end{aligned}$$

as $\mathrm{vol}(M) = \sin(\beta)^{n-2} \cdot \mathcal{O}_{n-2}$. Furthermore, from (11) and recalling $V_{-1}(K) = V_{n-1}(\breve{K})$, we get

$$V_{n-1}(K) = \frac{\mathcal{O}_{n-2}}{\mathcal{O}_{n-1}} \cdot \int_0^\beta \sin(\rho)^{n-2}\, d\rho \;, \qquad V_{-1}(K) = \frac{\mathcal{O}_{n-2}}{\mathcal{O}_{n-1}} \cdot \int_0^\beta \cos(\rho)^{n-2}\, d\rho \;.$$

We will finish this section by stating formulas for the intrinsic volumes of the cone of positive semidefinite matrices. These formulas have been developed in [3], and a paper containing this derivation is in preparation [5]. We include these formulas here so that we may appropriately formulate Conjecture 2.18 about the orders of magnitude of these intrinsic volumes.

For $z \in \mathbb{R}^k$ we denote the Vandermonde determinant by $\Delta(z) := \prod_{1 \le i < j \le k} (z_i - z_j)$. For $0 \le r \le k$ the Vandermonde determinant can be decomposed into

$$\Delta(z) = \Delta(x) \cdot \Delta(y) \cdot \sum_{\ell=0}^{r\cdot(k-r)} (-1)^{k-\ell} \cdot p_{r,\ell}(z) \;, \qquad p_{r,\ell}(z) := \sigma_\ell(x \otimes y^{-1}) \cdot \prod_{i=1}^{k-r} y_i^r \;,$$

with $x := (z_1, \ldots, z_r)$, $y := (z_{r+1}, \ldots, z_k)$, σ_ℓ denoting the ℓth elementary symmetric function, and

$$x \otimes y^{-1} := \left(\frac{x_1}{y_1}, \ldots, \frac{x_r}{y_1}, \frac{x_1}{y_2}, \ldots, \frac{x_r}{y_2}, \ldots, \frac{x_1}{y_{k-r}}, \ldots, \frac{x_r}{y_{k-r}} \right) \in \mathbb{R}^{r\cdot(k-r)} \;.$$

Lastly, we define for $0 \le r \le k$, $0 \le \ell \le r(k-r)$, and x, y as above,

$$J(k,r,\ell) \;:=\; \int\limits_{z \in \mathbb{R}_+^k} e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(x)| \cdot |\Delta(y)| \cdot p_{r,\ell}(z)\; dz \;,$$

and we set $J(k,r,\ell) := 0$ if the above inequalities on r and ℓ are not satisfied.

Theorem 2.13. *The intrinsic volumes of the cone of positive semidefinite matrices are given by*

$$V_{j-1}(\mathrm{Sym}_+^k) = \frac{1}{k! \cdot 2^{\frac{k}{2}} \cdot \prod_{d=1}^k \Gamma(\frac{d}{2})} \cdot \sum_{r=0}^{k} \binom{k}{r} \cdot J(k, r, j - t(r)) \;,$$

where $t(r) := \frac{r(r+1)}{2} = \dim(\mathrm{Sym}^r)$.

See the Figures 1,2 for graphical displays of the (logarithms of the) intrinsic volumes of Sym_+^k for $k = 4, 5, 6$. The values of the intrinsic volumes were obtained by numerically approximating the integrals $J(k, r, \ell)$ via a Monte-Carlo method. This explains the observable inaccuracies for $k = 6$.

2.3 Estimating intrinsic volumes

In this section we will describe some quantities, which are related to binomial coefficients and which provide a method to estimate the intrinsic volumes of self-dual cones. Recall that in (9) we encountered the volume of the unit sphere, which we denote by $\mathcal{O}_{n-1} = \mathrm{vol}_{n-1} S^{n-1}$. We denote the volume of the nth unit ball $B_n \subset \mathbb{R}^n$ by

$$\omega_n \;:=\; \mathrm{vol}_n B_n \;=\; \frac{\mathcal{O}_{n-1}}{n} \;=\; \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n+2}{2})} \quad \text{for } n > 0, \text{ and } \omega_0 := 1 \;. \tag{20}$$

It is convenient to use the analytic extension of the binomial coefficient. For $x > -1$ and $-1 < y < x + 1$ we denote $\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\cdot\Gamma(x-y+1)}$. In particular, for half-integers, i.e., for $n, m \in \mathbb{Z}$, $-1 \leq m \leq n + 1$, we have

$$\binom{n/2}{m/2} = \frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{m+2}{2}) \cdot \Gamma(\frac{n-m+2}{2})} \;. \tag{21}$$

Note that $\binom{n/2}{(n-m)/2} = \binom{n/2}{m/2}$ and $\binom{(n-2)/2}{-1/2} = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\cdot\Gamma(\frac{n+1}{2})}$.

Besides the binomial coefficient we also use the *flag coefficients* $\begin{bmatrix} n \\ m \end{bmatrix}$ as defined in [34]. These are given by

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+1}{2}) \cdot \Gamma(\frac{n-m+1}{2})} \;. \tag{22}$$

Note that $\begin{bmatrix} n \\ n-m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}$. The following proposition provides some useful identities involving the binomial and the flag coefficients.

Proposition 2.14. *1. For $n, m \in \mathbb{N}$, $n \geq m$,*

$$\binom{n}{m} = \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2}) \cdot \Gamma(\frac{n+2}{2})}{\Gamma(\frac{m+1}{2}) \cdot \Gamma(\frac{m+2}{2}) \cdot \Gamma(\frac{n-m+1}{2}) \cdot \Gamma(\frac{n-m+2}{2})} = \begin{bmatrix} n \\ m \end{bmatrix} \cdot \binom{n/2}{m/2} \;. \tag{23}$$

2. For $n, m \in \mathbb{N}$, $n \geq m$,

$$\binom{n/2}{m/2} = \frac{\omega_m \cdot \omega_{n-m}}{\omega_n} \quad , \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{\mathcal{O}_m \cdot \mathcal{O}_{n-m}}{2 \cdot \mathcal{O}_n} \;. \tag{24}$$

In particular,

$$\binom{n}{m} \cdot \frac{\omega_n}{\omega_m \cdot \omega_{n-m}} = \begin{bmatrix} n \\ m \end{bmatrix} \quad , \qquad \binom{n}{m} \cdot \frac{2 \cdot \mathcal{O}_n}{\mathcal{O}_m \cdot \mathcal{O}_{n-m}} = \binom{n/2}{m/2} . \tag{25}$$

3. *For $n, m \to \infty$ such that also $(n-m) \to \infty$,*

$$\begin{bmatrix} n \\ m \end{bmatrix} \sim \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{m(n-m)}{n}} \cdot \binom{n/2}{m/2} , \tag{26}$$

where the symbol $\sim$ means that the quotient of the two sides tends to one.

Proof. The first equation in (23) follows from applying the duplication formula of the Γ-function $\Gamma(2x) = \frac{1}{\sqrt{\pi}} \cdot 2^{2x-1} \cdot \Gamma(x) \cdot \Gamma(x + \frac{1}{2})$ to the term $\binom{n}{m} = \frac{\Gamma(n+1)}{\Gamma(m+1) \cdot \Gamma(n-m+1)}$. The remaining equations follow by plugging in the definitions of the corresponding quantities. As for the asymptotics stated in (26), we compute

$$\begin{aligned} \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{m(n-m)}{n}} \cdot \binom{n/2}{m/2} &= \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{m(n-m)}{n}} \cdot \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{m}{2}+1) \cdot \Gamma(\frac{n-m}{2}+1)} \\ &= \frac{\sqrt{\pi} \cdot \sqrt{\frac{n}{2}} \cdot \Gamma(\frac{n}{2})}{\sqrt{\frac{m}{2}} \cdot \Gamma(\frac{m}{2}) \cdot \sqrt{\frac{n-m}{2}} \cdot \Gamma(\frac{n-m}{2})} \sim \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2})}{\Gamma(\frac{m+1}{2}) \cdot \Gamma(\frac{n-m+1}{2})} = \begin{bmatrix} n \\ m \end{bmatrix} , \end{aligned}$$

where we have used the asymptotics $\sqrt{x} \cdot \Gamma(x) \sim \Gamma(x + \frac{1}{2})$ for $x \to \infty$. □

With this notation we can revisit Example 2.12 and improve the formulas for the intrinsic volumes of circular caps.

Example 2.15. Let $K = B(z, \beta)$, $0 < \beta \leq \pi/2$, be a circular cap. Then from Example 2.12 we have for $1 \leq j \leq n-1$

$$\begin{aligned} V_{j-1}(K) &= \frac{\mathcal{O}_{n-2}}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{n-j-1}} \cdot \binom{n-2}{j-1} \cdot \sin(\beta)^{j-1} \cdot \cos(\beta)^{n-j-1} \\ &\overset{(25)}{=} \binom{(n-2)/2}{(j-1)/2} \cdot \frac{\sin(\beta)^{j-1} \cdot \cos(\beta)^{n-j-1}}{2} . \end{aligned}$$

Furthermore, we have for $j = n$

$$V_{n-1}(K) \overset{(11)}{=} \frac{\mathcal{O}_{n-2}}{\mathcal{O}_{n-1}} \cdot \int_0^\beta \sin(\rho)^{n-2} \, d\rho \overset{(21)}{=} \binom{(n-2)/2}{(n-1)/2} \cdot \frac{n-1}{2} \cdot \int_0^\beta \sin(\rho)^{n-2} \, d\rho ,$$

and similarly for $j = 0$

$$V_{-1}(K) = \binom{(n-2)/2}{-1/2} \cdot \frac{n-1}{2} \cdot \int_0^{\frac{\pi}{2}-\beta} \sin(\rho)^{n-2} \, d\rho .$$

For Lorentz caps, i.e., $\beta = \frac{\pi}{4}$, we have $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$, and the formulas for $V_{j-1}(\mathcal{L}^n)$, which we denote by $f_j(n)$ for convenience, simplify to

$$f_j(n) := V_{j-1}(\mathcal{L}^n) = \frac{\binom{(n-2)/2}{(j-1)/2}}{2^{n/2}} \;, \quad \text{for } 1 \le j \le n-1 \;, \tag{27}$$

$$\begin{aligned} f_0(n) := f_n(n) := V_{-1}(\mathcal{L}^n) = V_{n-1}(\mathcal{L}^n) &= \frac{\binom{(n-2)/2}{-1/2}}{2^{n/2}} \cdot 2^{(n-2)/2} \cdot (n-1) \cdot \int_0^{\pi/4} \sin(\rho)^{n-2}\, d\rho \\ &= \frac{\binom{(n-2)/2}{-1/2}}{2^{n/2}} \cdot {}_2F_1\left(1, \tfrac{1}{2}; \tfrac{n+1}{2}; -1\right) \quad \left[\sim \frac{\binom{(n-2)/2}{-1/2}}{2^{n/2}} \;, \text{ for } n \to \infty\right] . \end{aligned}$$

where ${}_2F_1$ denotes the ordinary hypergeometric function (cf. [1, Ch. 15]). Note that the sequence $f(n)$ is symmetric, i.e., $f_{n-j}(n) = f_j(n)$.

It is convenient to estimate the intrinsic volumes of self-dual cones by means of the intrinsic volumes of the Lorentz cones. We therefore define for a self-dual cone $C \subseteq \mathbb{R}^n$ the *excess over the Lorentz cone* $v(C)$ as

$$v(C) := \min_{0 \le j \le n} \frac{V_{j-1}(C)}{f_j(n)} \;. \tag{28}$$

In other words, $v(C)$ is the smallest constant such that the inequality $V(C) \le v(C) \cdot f(n)$ is satisfied componentwise. The constant $v(C)$ will come in handy for the estimations, as we will see in Section 3.

Proposition 2.16. *For the positive orthant and for the Lorentz cone we have*

$$v(\mathbb{R}^n_+) < \sqrt{2} \;, \qquad v(\mathcal{L}^n) = 1 \;.$$

Proof. The claim about the Lorentz cone follows trivially from the definition of v and f. As for the positive orthant, we use Example 2.10 and compute for $1 \le j \le n-1$

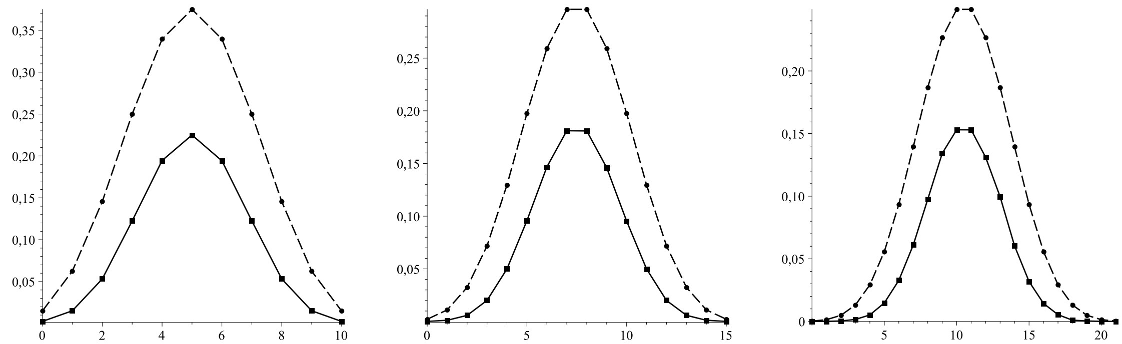
$$\begin{aligned} \frac{V_{j-1}(\mathbb{R}^n_+)}{f_j(n)} &= \frac{\binom{n}{j}}{\binom{(n-2)/2}{(j-1)/2} \cdot 2^{n/2}} \overset{(21),\,(23)}{=} \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2}) \cdot \Gamma(\frac{n+2}{2}) \cdot \Gamma(\frac{j+1}{2}) \cdot \Gamma(\frac{n-j+1}{2})}{\Gamma(\frac{j+1}{2}) \cdot \Gamma(\frac{j+2}{2}) \cdot \Gamma(\frac{n-j+1}{2}) \cdot \Gamma(\frac{n-j+2}{2}) \cdot \Gamma(\frac{n}{2})} \cdot \frac{1}{2^{n/2}} \\ &\overset{(21)}{=} \sqrt{\pi} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\binom{n/2}{j/2}}{2^{n/2}} < \sqrt{\pi} \cdot \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})} \cdot \frac{\binom{n/2}{n/4}}{2^{n/2}} < \sqrt{2} \;, \end{aligned}$$

where the last inequality is easily checked with a computer algebra system. As for $j \in \{0, n\}$, we have

$$\frac{V_{n-1}(\mathbb{R}^n_+)}{f_n(n)} = \frac{V_{-1}(\mathbb{R}^n_+)}{f_0(n)} = \frac{1}{2^{n/2}} \cdot \frac{\sqrt{\pi} \cdot \Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}) \cdot {}_2F_1\left(1, \frac{1}{2}; \frac{n+1}{2}; -1\right)} \le 1 \;,$$

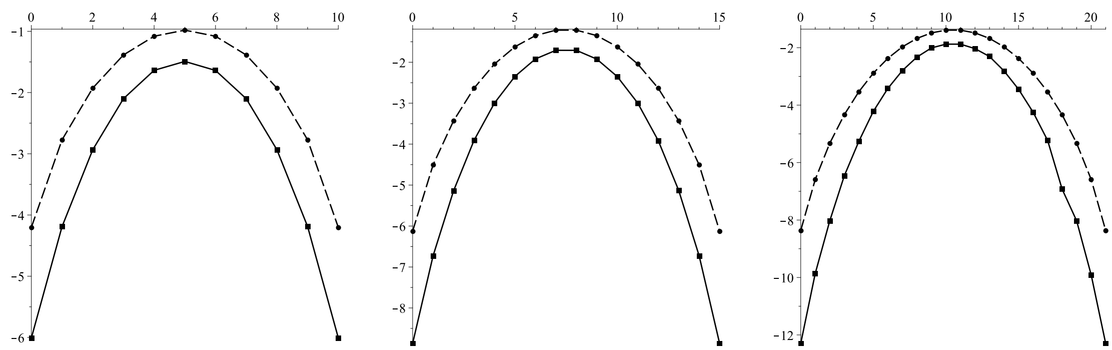
where again the last inequality is easily checked with a computer algebra system. □

An important construction in convex optimization is taking direct products. It is thus desirable to have an easy estimate for $v(C_1 \times C_2)$. Unfortunately, this is more challenging than it might appear at first sight. The following conjecture is supported both by geometric reasonings and computer-aided experiments.



(a) $k = 4$ (b) $k = 5$ (c) $k = 6$

Figure 1: The intrinsic volumes of Sym_+^k (solid) and the upper bound $2 \cdot f(t(k))$ (dashed), illustrating Conjecture 2.18.



(a) $k = 4$ (b) $k = 5$ (c) $k = 6$

Figure 2: The logarithms of the intrinsic volumes of Sym_+^k (solid) and the logarithm of the upper bound $2 \cdot f(t(k))$ (dashed), illustrating Conjecture 2.18.

Conjecture 2.17. For $n_1, n_2 \in \mathbb{N}$ we have

$$f(n_1) * f(n_2) < 2 \cdot f(n_1 + n_2) \ . \tag{29}$$

In particular, for closed convex cones C_1, C_2 we have $v(C_1 \times C_2) \leq 2 \cdot v(C_1) \cdot v(C_2)$, and thus

$$v(\mathcal{L}^{n_1} \times \ldots \times \mathcal{L}^{n_k}) < 2^{k-1} \ . \tag{30}$$

For the cone of positive semidefinite matrices we put up the following conjecture about the order of magnitude of its intrinsic volumes.

Conjecture 2.18. For the cone Sym_+^k of positive semidefinite matrices we have

$$v(\mathrm{Sym}_+^k) < 2 \ . \tag{31}$$

See Figure 1 and Figure 2 for an illustration of this conjecture.

The constant 2 in (29) and (31) is in both cases probably not optimal. In fact, concerning (29), computations suggest that $f(n_1) * f(n_2) = (1+o(1)) \cdot f(n_1+n_2)$ for $n_1, n_2 \to \infty$. Our conjectures about the excess over the Lorentz cone are succinctly summarized in the table of Theorem 1.4.

Concluding this section, we point out that the statistical behavior of intrinsic volumes is currently a field with a wealth of open questions. Note that Figure 1 shows an apparently unimodal sequence, and Figure 2 suggests that it should actually be log-concave. We conjecture that this holds in general.

Conjecture 2.19. For every $K \in \mathcal{K}(S^{n-1})$ the sequence of intrinsic volumes $V(K)$ is *log-concave*, i.e., $V_j(K)^2 \geq V_{j-1}(K) \cdot V_{j+1}(K)$ for $0 \leq j \leq n-2$.

Remark 2.20. Using the formulas in Example 2.15 and the calculation rule from Corollary 2.9, one can show (cf. [3, Cor. 4.4.14]) that a direct product of circular cones has a log-concave sequence of intrinsic volumes. In particular, the intrinsic volumes of the positive orthant and of products of Lorentz cones are log-concave. If Conjecture 2.19 is true, it could be considered as a spherical analog of the famous *Alexandrov-Fenchel inequalities* (cf. [52, 47]).

3 A tube formula for the Grassmann manifold

Before we state the main theorem we introduce the following notation for a rescaling of the volume of the tube of radius α around an i-dimensional subsphere (cf. (10))

$$I_{n,i}(\alpha) := \frac{\mathcal{O}_{n-1,i}(\alpha)}{\mathcal{O}_i \cdot \mathcal{O}_{n-2-i}} \overset{(10)}{=} \int_0^\alpha \cos(\rho)^i \cdot \sin(\rho)^{n-2-i}\, d\rho = \int_0^\tau \frac{t^{n-2-i}}{(1+t^2)^{n/2}}\, dt \ , \tag{32}$$

where $\tau := \tan\alpha$.

In the following, we fix a regular cone $C \subset \mathbb{R}^n$ and set $\mathcal{D}_m := \mathcal{D}_m(C)$, $\mathcal{P}_m := \mathcal{D}_m(C)$, and $\Sigma_m := \Sigma_m(C)$, cf. Definition 1.1. Furthermore, we define the *tube around* Σ_m as

$$\mathcal{T}(\Sigma_m, \alpha) := \{W \in \mathrm{Gr}_{n,m} \mid \exists W' \in \Sigma_m : d_g(W, W') \leq \alpha\} \ , \tag{33}$$

where d_g denotes the geodesic distance in $\mathrm{Gr}_{n,m}$ (see [4] for more details). We also define the *primal* and the *dual tube around* Σ_m, respectively, as

$$\mathcal{T}^{\mathrm{P}}(\Sigma_m, \alpha) := \mathcal{T}(\Sigma_m, \alpha) \cap \mathcal{P}_m \ , \qquad \mathcal{T}^{\mathrm{D}}(\Sigma_m, \alpha) := \mathcal{T}(\Sigma_m, \alpha) \cap \mathcal{D}_m \ . \tag{34}$$

In the following theorem we will give an estimate of the *relative volume* rvol of the primal tube around Σ_m. By the relative volume we mean the unique probability measure on the homogeneous space $\mathrm{Gr}_{n,m}$, which is invariant under the action of the orthogonal group (see Section 5 for more details on this). The proof of Theorem 3.1 below is postponed to Section 6.

Theorem 3.1. *Let $C \subset \mathbb{R}^n$ be a regular cone. Then, for $1 \leq m \leq n-1$ and $0 \leq \alpha \leq \frac{\pi}{2}$,*

$$\mathrm{rvol}\ \mathcal{T}^{\mathrm{P}}(\Sigma_m(C), \alpha) \leq \frac{2m(n-m)}{n} \binom{n/2}{m/2} \cdot \sum_{j=0}^{n-2} V_j(C) \cdot \begin{bmatrix} n-2 \\ j \end{bmatrix} \cdot \sum_{i=0}^{n-2} d_{ij}^{nm} \cdot I_{n,i}(\alpha)\,, \tag{35}$$

$$
D_{7,1} = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}, \quad
D_{7,2} = \begin{pmatrix} & \frac{1}{5} &&&& \\ 1 & 0 & \frac{2}{5} &&& \\ & \frac{4}{5} & 0 & \frac{3}{5} && \\ && \frac{3}{5} & 0 & \frac{4}{5} & \\ &&& \frac{2}{5} & 0 & 1 \\ &&&& \frac{1}{5} & \end{pmatrix}, \quad
D_{7,3} = \begin{pmatrix} && \frac{1}{10} &&& \\ & \frac{2}{5} & 0 & \frac{3}{10} && \\ 1 & 0 & \frac{3}{5} & 0 & \frac{3}{5} & \\ & \frac{3}{5} & 0 & \frac{3}{5} & 0 & 1 \\ && \frac{3}{10} & 0 & \frac{2}{5} & \\ &&& \frac{1}{10} && \end{pmatrix},
$$

$$
D_{7,6} = \begin{pmatrix} &&&&& 1 \\ &&&& 1 & \\ &&& 1 && \\ && 1 &&& \\ & 1 &&&& \\ 1 &&&&& \end{pmatrix}, \quad
D_{7,5} = \begin{pmatrix} &&&& \frac{1}{5} & \\ &&& \frac{2}{5} & 0 & 1 \\ && \frac{3}{5} & 0 & \frac{4}{5} & \\ & \frac{4}{5} & 0 & \frac{3}{5} && \\ 1 & 0 & \frac{2}{5} &&& \\ & \frac{1}{5} &&&& \end{pmatrix}, \quad
D_{7,4} = \begin{pmatrix} &&& \frac{1}{10} && \\ && \frac{3}{10} & 0 & \frac{2}{5} & \\ & \frac{3}{5} & 0 & \frac{3}{5} & 0 & 1 \\ 1 & 0 & \frac{3}{5} & 0 & \frac{3}{5} & \\ & \frac{2}{5} & 0 & \frac{3}{10} && \\ && \frac{1}{10} &&& \end{pmatrix}.
$$

Table 1: The coefficient matrices $D_{n,m} = (d_{ij}^{nm})_{i,j=0,\ldots,n-2}$ for $n = 7$, $m = 1, \ldots, 6$.

where the constants d_{ij}^{nm} are defined for $i + j + m \equiv 1 \pmod 2$, $0 \leq \frac{i-j}{2} + \frac{m-1}{2} \leq m - 1$, and $0 \leq \frac{i+j}{2} - \frac{m-1}{2} \leq n - m - 1$, via

$$
d_{ij}^{nm} \;:=\; \frac{\binom{m-1}{\frac{i-j}{2}+\frac{m-1}{2}} \cdot \binom{n-m-1}{\frac{i+j}{2}-\frac{m-1}{2}}}{\binom{n-2}{j}}\,, \tag{36}
$$

and $d_{ij}^{nm} := 0$ otherwise.

Remark 3.2.
- It can be shown (see [3] for details) that (35) in fact holds *with equality* if $\mathcal{T}(C \cap S^{n-1}, \alpha)$ is convex and if d_{ij}^{nm} is replaced by $(-1)^{\frac{i-j}{2}-\frac{m-1}{2}} \cdot d_{ij}^{nm}$. This reveals how close is our estimate (35) to being sharp.
- The coefficients d_{ij}^{nm} satisfy the symmetry relations

$$
d_{i,n-2-j}^{n,n-m} = d_{ij}^{nm}\,, \qquad d_{n-2-i,n-2-j}^{nm} = d_{ij}^{nm}\,. \tag{37}
$$

See Table 1 for a display of the coefficient matrix $D_{n,m} = (d_{ij}^{nm})_{i,j=0,\ldots,n-2}$.

The following corollary will be obtained from Theorem 3.1 by using duality properties as described in [4] to obtain an estimate of the dual tube of Σ_m, and by changing the summation over i, j as depicted in Figure 3.

Corollary 3.3. *Let $C \subset \mathbb{R}^n$ be a regular cone and let d_{ij}^{nm} be defined as in (36). Then, for $1 \leq m \leq n-1$ and $0 \leq \alpha \leq \frac{\pi}{2}$, and using the convetion $\binom{k}{\ell} := 0$ if $\ell < 0$ or $\ell > k$, we have*

$$
\begin{aligned}
\operatorname{rvol} \mathcal{T}(\Sigma_m(C), \alpha) \leq 8 \cdot \sum_{i,k=0}^{n-2} V_{n-m-i+2k-1}(C) \cdot \frac{\Gamma(\frac{m+i-2k+1}{2})}{\Gamma(\frac{m}{2})} \cdot \frac{\Gamma(\frac{n-m-i+2k+1}{2})}{\Gamma(\frac{n-m}{2})} \\
\cdot \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} \cdot I_{n,n-2-i}(\alpha)\,.
\end{aligned} \tag{38}
$$

Proof. The involution $\iota\colon \mathrm{Gr}_{n,m} \to \mathrm{Gr}_{n,n-m}$, $\mathcal{W} \mapsto \mathcal{W}^\perp$ is an isometry and thus preserves the geodesic distance. Moreover, ι maps $\mathcal{P}_m(C)$ to $\mathcal{D}_{n-m}(\breve{C})$, and therefore $\iota(\Sigma_m(C)) = \Sigma_{n-m}(\breve{C})$, cf. [4] . Hence, ι induces a bijection between $\mathcal{T}^{\mathrm{D}}(\Sigma_m(C),\alpha)$ and $\mathcal{T}^{\mathrm{P}}(\Sigma_{n-m}(\breve{C}),\alpha)$. So we get

$$\begin{aligned}\mathrm{rvol}\ \mathcal{T}(\Sigma_m(C),\alpha) &\overset{(34)}{=} \mathrm{rvol}\ \mathcal{T}^{\mathrm{P}}(\Sigma_m(C),\alpha) + \mathrm{rvol}\ \mathcal{T}^{\mathrm{D}}(\Sigma_m(C),\alpha)\\ &= \mathrm{rvol}\ \mathcal{T}^{\mathrm{P}}(\Sigma_m(C),\alpha) + \mathrm{rvol}\ \mathcal{T}^{\mathrm{P}}(\Sigma_{n-m}(\breve{C}),\alpha)\ ,\end{aligned}$$

From Theorem 3.1 we get, using the duality property $V_j(\breve{C}) = V_{n-2-j}(C)$ (cf. (17)) and the symmetry relation $d^{n,n-m}_{i,n-2-j} = d^{nm}_{ij}$ (cf. (37)),

$$\mathrm{rvol}\ \mathcal{T}(\Sigma_m(C),\alpha) \le \frac{4m(n-m)}{n}\binom{n/2}{m/2}\cdot\sum_{j=0}^{n-2} V_j(C)\cdot\begin{bmatrix} n-2\\ j\end{bmatrix}\cdot\sum_{i=0}^{n-2} d^{nm}_{ij}\cdot I_{n,i}(\alpha)\ . \qquad (39)$$

Using Proposition 2.14, we get

$$\begin{aligned}&\frac{4m(n-m)}{n}\cdot\frac{\binom{n/2}{m/2}\cdot\begin{bmatrix} n-2\\ j\end{bmatrix}}{\binom{n-2}{j}} \overset{(23)}{=} \frac{4m(n-m)}{n}\cdot\frac{\binom{n/2}{m/2}}{\binom{(n-2)/2}{j/2}}\\ &\overset{(21)}{=} \frac{4m(n-m)}{n}\cdot\frac{\Gamma(\frac{n+2}{2})}{\Gamma(\frac{m+2}{2})\cdot\Gamma(\frac{n-m+2}{2})}\cdot\frac{\Gamma(\frac{j+2}{2})\cdot\Gamma(\frac{n-j}{2})}{\Gamma(\frac{n}{2})} = 8\cdot\frac{\Gamma(\frac{j+2}{2})}{\Gamma(\frac{m}{2})}\cdot\frac{\Gamma(\frac{n-j}{2})}{\Gamma(\frac{n-m}{2})}\ .\end{aligned}$$

Using this in (39), changing the summation via $i \leftarrow n-2-i$ and $j \leftarrow n-2-j$, and taking into account (37), we thus get

$$\begin{aligned}\mathrm{rvol}\ \mathcal{T}(\Sigma_m(C),\alpha) &\overset{(39)}{\le} 8\cdot\sum_{i,j=0}^{n-2}\frac{\Gamma(\frac{j+2}{2})}{\Gamma(\frac{m}{2})}\cdot\frac{\Gamma(\frac{n-j}{2})}{\Gamma(\frac{n-m}{2})}\cdot V_j(C)\cdot\binom{n-2}{j}\cdot d^{nm}_{ij}\cdot I_{n,i}(\alpha)\\ &\overset{(37)}{=} 8\cdot\sum_{i,j=0}^{n-2}\frac{\Gamma(\frac{j+2}{2})}{\Gamma(\frac{m}{2})}\cdot\frac{\Gamma(\frac{n-j}{2})}{\Gamma(\frac{n-m}{2})}\cdot V_{n-2-j}(C)\cdot\binom{n-2}{j}\cdot d^{nm}_{ij}\cdot I_{n,n-2-i}(\alpha) \qquad (40)\\ &= 8\cdot\sum_{\substack{i,j=0\\ i+j+m\equiv 1\\ (\mathrm{mod}\ 2)}}^{n-2} V_{n-2-j}(C)\cdot\frac{\Gamma(\frac{j+2}{2})}{\Gamma(\frac{m}{2})}\cdot\frac{\Gamma(\frac{n-j}{2})}{\Gamma(\frac{n-m}{2})}\cdot\binom{m-1}{\frac{i-j}{2}+\frac{m-1}{2}}\cdot\binom{n-m-1}{\frac{i+j}{2}-\frac{m-1}{2}}\cdot I_{n,n-2-i}(\alpha)\ .\end{aligned}$$

Here we interpret $\binom{k}{\ell} = 0$ if $\ell < 0$ or $\ell > k$, i.e., the above summation over i,j in fact only runs over the rectangle determined by the inequalities $0 \le \frac{i-j}{2} + \frac{m-1}{2} \le m-1$ and $0 \le \frac{i+j}{2} - \frac{m-1}{2} \le n-m-1$ (cf. Figure 3). As the summation runs only over those i,j, for which $i+j+m \equiv 1 \bmod 2$, we may replace the summation over j by a summation over $k = \frac{i-j}{2} + \frac{m-1}{2}$. The above inequalities then transform into $0 \le k \le m-1$ and $0 \le i-k \le n-m-1$. So we get from (40)

$$\begin{aligned}\mathrm{rvol}\ \mathcal{T}(\Sigma_m(C),\alpha) = 8\cdot\sum_{i,k=0}^{n-2} V_{n-m-1-i+2k}(C)\cdot\frac{\Gamma(\frac{m+i-2k+1}{2})}{\Gamma(\frac{m}{2})}\cdot\frac{\Gamma(\frac{n-m-i+2k+1)}{2})}{\Gamma(\frac{n-m}{2})}\\ \cdot\binom{m-1}{k}\cdot\binom{n-m-1}{i-k}\cdot I_{n,n-2-i}(\alpha)\ . \qquad \square\end{aligned}$$

Before giving the proofs of the main Theorems 1.3 and 1.4 by estimating the tube formula (38), we will provide a number of technical estimates in the following lemma. In the proof and in the rest of this section we will mark estimates, which are easily checked with a computer algebra system, with the symbol 🖩.

Lemma 3.4. *Let $i, k, \ell, m, n \in \mathbb{N}$ with $n \geq 2$ and $1 \leq m \leq n-1$.*

1. *We have*
$$\frac{\Gamma(\frac{m+\ell+1}{2})}{\Gamma(\frac{m}{2})} \leq \sqrt{\frac{m}{2}} \cdot \left(\frac{m+\ell}{2}\right)^{\frac{\ell}{2}} . \tag{41}$$

2. *For $0 \leq k \leq m-1$ and $0 \leq i-k \leq n-m-1$ we have*
$$\left(\frac{m+i-2k}{n-m-i+2k}\right)^{\frac{i-2k}{2}} < n^{\frac{i}{2}} . \tag{42}$$

3. *For $0 \leq \alpha \leq \frac{\pi}{2}$, $t := \sin(\alpha)^{-1}$, and $n \geq 3$, we have*
$$\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot n^{\frac{i}{2}} \cdot I_{n,n-2-i}(\alpha) \;<\; \frac{3}{t}\,, \quad \textit{if } t > n^{\frac{3}{2}}\,, \tag{43}$$
$$\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot I_{n,n-2-i}(\alpha) \;<\; \exp\left(\frac{n}{m}\right) \cdot \frac{1}{t}\,, \quad \textit{if } t > m\,. \tag{44}$$

Proof. (1) We make a case distinction by the parity of ℓ. Using $\Gamma(x+1) = x \cdot \Gamma(x)$, we get for odd ℓ
$$\frac{\Gamma(\frac{m+\ell+1}{2})}{\Gamma(\frac{m}{2})} = \prod_{a=0}^{\frac{\ell-1}{2}} \left(\frac{m}{2} + a\right) \leq \frac{m}{2} \cdot \left(\frac{m+\ell-1}{2}\right)^{\frac{\ell-1}{2}} \leq \sqrt{\frac{m}{2}} \cdot \left(\frac{m+\ell}{2}\right)^{\frac{\ell}{2}} .$$
Using additionally $\Gamma(x+\frac{1}{2}) < \sqrt{x} \cdot \Gamma(x)$, we get for even ℓ
$$\frac{\Gamma(\frac{m+\ell+1}{2})}{\Gamma(\frac{m}{2})} = \prod_{a=0}^{\frac{\ell}{2}-1} \left(\frac{m+1}{2} + a\right) \cdot \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} < \left(\frac{m+\ell}{2}\right)^{\frac{\ell}{2}} \cdot \sqrt{\frac{m}{2}} .$$

(2) As for the second estimate, we distinguish the cases $i \geq 2k$ and $i \leq 2k$. From $0 \leq k \leq m-1$ and $0 \leq i-k \leq n-m-1$ we get
$$\begin{aligned} 1 \;\leq\; & \quad m-k+i-k & \leq\; n-1\,, \\ 1 \;\leq\; & n-(m-k+i-k) & \leq\; n-1\,. \end{aligned}$$
For $i \geq 2k$ we thus get
$$\left(\frac{m+i-2k}{n-m-i+2k}\right)^{\frac{i-2k}{2}} \leq (n-1)^{\frac{i-2k}{2}} < n^{\frac{i}{2}},$$

and for $i \leq 2k$

$$\left(\frac{m+i-2k}{n-m-i+2k}\right)^{\frac{i-2k}{2}} = \left(\frac{n-m+2k-i}{m-2k+i}\right)^{\frac{2k-i}{2}} \leq (n-1)^{\frac{2k-i}{2}} < n^{\frac{i}{2}} \ .$$

(3) The I-functions have been estimated in [14, Lemma 2.2] in the following way. Let $\varepsilon := \sin(\alpha) = \frac{1}{t}$. For $i < n-2$

$$I_{n,n-2-i}(\alpha) = \int_0^\alpha \cos(\rho)^{n-2-i} \cdot \sin(\rho)^i \, d\rho \leq \frac{\varepsilon^{i+1}}{i+1} \ ,$$

and for $i = n-2$, assuming $n \geq 3$,

$$I_{n,0}(\alpha) = \int_0^\alpha \sin(\rho)^{n-2} \, d\rho \leq \frac{\mathcal{O}_{n-1} \cdot \varepsilon^{n-1}}{2\mathcal{O}_{n-2}} = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \cdot \varepsilon^{n-1} < \sqrt{\frac{\pi}{2(n-2)}} \cdot \varepsilon^{n-1} \ .$$

With these estimates we get

$$\begin{aligned}
&\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot n^{\frac{i}{2}} \cdot I_{n,n-2-i}(\alpha) \\
&\qquad \leq \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot n^{\frac{i}{2}} \cdot \frac{\varepsilon^{i+1}}{i+1} + \left(\sqrt{\frac{\pi}{2(n-2)}} - \frac{1}{n-1}\right) \cdot \varepsilon^{n-1} \cdot n^{\frac{n-2}{2}} \\
&\qquad < \varepsilon \cdot \left(\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot n^{\frac{i}{2}} \cdot \varepsilon^i + \frac{1.4}{\sqrt{n}} \cdot \varepsilon^{n-2} \cdot n^{\frac{n-2}{2}}\right) \\
&\qquad = \varepsilon \cdot \left(\left(1 + \sqrt{n} \cdot \varepsilon\right)^{n-2} + 1.4 \cdot \varepsilon^{n-2} \cdot n^{\frac{n-3}{2}}\right) .
\end{aligned}$$

For $\varepsilon < n^{-\frac{3}{2}}$ we thus get

$$\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot n^{\frac{i}{2}} \cdot I_{n,n-2-i}(\alpha) < \varepsilon \cdot \left(\underbrace{\left(1 + \frac{1}{n}\right)^{n-2}}_{<\exp(1)} + 1.4 \cdot n^{\frac{3}{2}-n}\right) < 3 \cdot \varepsilon \ .$$

Similarly we derive

$$\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot I_{n,n-2-i}(\alpha) < \varepsilon \cdot \left((1+\varepsilon)^{n-2} + \frac{1.4}{\sqrt{n}} \cdot \varepsilon^{n-2}\right) \ .$$

For $\varepsilon < \frac{1}{m}$ we thus get

$$\sum_{i=0}^{n-2} \binom{n-2}{i} \cdot I_{n,n-2-i}(\alpha) < \varepsilon \cdot \left(\left(1 + \frac{1}{m}\right)^{n-2} + \frac{1.4}{\sqrt{n} \cdot m^{n-2}}\right) < \varepsilon \cdot \exp\left(\frac{n}{m}\right) \ . \qquad \square$$

Proof of Theorem 1.3. If C is a regular cone then it is not a linear subspace. The intrinsic volumes are therefore bounded by $V_j(C) \leq \frac{1}{2}$ (cf. Remark 2.8). Using Lemma 3.4 we get from Corollary 3.3

$$
\begin{aligned}
\operatorname{rvol} \mathcal{T}(\Sigma_m, \alpha) &\overset{(38)}{\leq} 8 \cdot \sum_{i,k=0}^{n-2} V_{n-m-i+2k-1}(C) \cdot \frac{\Gamma(\frac{m+i-2k+1}{2})}{\Gamma(\frac{m}{2})} \cdot \frac{\Gamma(\frac{n-m-i+2k+1}{2})}{\Gamma(\frac{n-m}{2})} \\
&\qquad\qquad \cdot \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} \cdot I_{n,n-2-i}(\alpha) \\
&\overset{(41)}{\leq} 2 \cdot \sqrt{m(n-m)} \cdot \sum_{i,k=0}^{n-2} \left(\frac{m+i-2k}{2}\right)^{\frac{i-2k}{2}} \cdot \left(\frac{n-m-i+2k}{2}\right)^{-\frac{i-2k}{2}} \\
&\qquad\qquad \cdot \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} \cdot I_{n,n-2-i}(\alpha) \\
&\overset{(42)}{\leq} 2 \cdot \sqrt{m(n-m)} \cdot \sum_{i=0}^{n-2} n^{\frac{i}{2}} \cdot I_{n,n-2-i}(\alpha) \cdot \sum_{k=0}^{n-2} \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} .
\end{aligned}
$$

By Vandermonde's identity we have $\sum_{k=0}^{n-2} \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} = \binom{n-2}{i}$. So we get

$$
\begin{aligned}
\operatorname{rvol} \mathcal{T}(\Sigma_m, \alpha) &\leq 2 \cdot \sqrt{m(n-m)} \cdot \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot n^{\frac{i}{2}} \cdot I_{n,n-2-i}(\alpha) \\
&\overset{(43)}{<} 6 \cdot \sqrt{m(n-m)} \cdot \frac{1}{t} , \quad \text{if } t > n^{\frac{3}{2}} \text{ and } n \geq 3 .
\end{aligned}
$$

As for the expectation of the logarithm of the Grassmann condition, we compute

$$
\begin{aligned}
\mathbb{E}\left[\ln(\mathscr{C}_{\mathrm{G}}(A))\right] &= \int_0^\infty \operatorname{Prob}[\ln(\mathscr{C}_{\mathrm{G}}(A)) > s]\, ds \\
&< 1.5 \cdot \ln(n) + r + \int_{\ln(n^{3/2})+r}^{\infty} 6 \cdot \sqrt{m(n-m)} \cdot \exp(-s)\, ds \\
&= 1.5 \cdot \ln(n) + r + \underbrace{6 \cdot \frac{\sqrt{m(n-m)}}{n^{3/2}}}_{\leq \frac{2}{3}\sqrt{6}\text{ , if } n \geq 3} \cdot \exp(-r) \\
&< 1.5 \cdot \ln(n) + 1.5 ,
\end{aligned}
$$

if we choose $r := \frac{1}{2} \ln\left(\frac{8}{3}\right)$. □

Before we finish this section with the proof of Theorem 1.4 we need yet another technical lemma. A sequence (a_n) of nonnegative real numbers is called *log-concave* iff $a_n^2 \geq a_{n-1} \cdot a_{n+1}$ for all n. See [52] for a survey on log-concave sequences and their appearances in diverse areas of mathematics.

Lemma 3.5. *For $n \geq 2$ and $1 \leq m \leq n-1$ let $g_m(n) := \frac{\Gamma(\frac{n}{2}) \cdot \exp(\frac{n}{m})}{\Gamma(\frac{m}{2}) \cdot \Gamma(\frac{n-m}{2}) \cdot 2^{n/2}}$.*

1. *The sequence $(g_m(n))_n$ is log-concave, i.e., $g_m(n)^2 \geq g_m(n-1) \cdot g_m(n+1)$ for $n \geq m+2$.*

2. For fixed $m \geq 8$ we have $\max\{g_m(n) \mid n > m\} = \max\{g_m(2m+k) \mid k \in \{5,6,7\}\}$.

3. For $m \geq 8$ we have

$$g_m(n) < 2.5 \cdot \sqrt{m}\ . \tag{45}$$

Proof. (1) We have

$$\frac{g_m(n)^2}{g_m(n-1)\cdot g_m(n+1)} = \frac{\Gamma(\frac{n}{2})^2}{\Gamma(\frac{n-m}{2})^2}\cdot\frac{\Gamma(\frac{n-1-m}{2})}{\Gamma(\frac{n-1}{2})}\cdot\frac{\Gamma(\frac{n+1-m}{2})}{\Gamma(\frac{n+1}{2})}\ .$$

In order to show that this expression is ≥ 1 we use induction on m. For $m = 0$ this is trivially true, and for $m = 1$ this is easily checked with a computer algebra system. For $m \geq 2$ we have, using $\Gamma(x+1) = x \cdot \Gamma(x)$,

$$\begin{aligned}&\frac{\Gamma(\frac{n}{2})^2}{\Gamma(\frac{n-m}{2})^2}\cdot\frac{\Gamma(\frac{n-1-m}{2})}{\Gamma(\frac{n-1}{2})}\cdot\frac{\Gamma(\frac{n+1-m}{2})}{\Gamma(\frac{n+1}{2})}\\ &= \underbrace{\frac{(n-m)^2}{(n-1-m)\cdot(n+1-m)}}_{>1}\cdot\underbrace{\frac{\Gamma(\frac{n}{2})^2}{\Gamma(\frac{n-(m-2)}{2})^2}\cdot\frac{\Gamma(\frac{n-1-(m-2)}{2})}{\Gamma(\frac{n-1}{2})}\cdot\frac{\Gamma(\frac{n+1-(m-2)}{2})}{\Gamma(\frac{n+1}{2})}}_{\geq 1 \text{ by ind. hyp.}} > 1\ .\end{aligned}$$

(2) As the sequence $(g_m(n))_n$ is log-concave and positive, it follows that it is *unimodal*. This means that there exists an index N such that $g_m(n-1) \leq g_m(n)$ for all $n \leq N$, and $g_m(n) \geq g_m(n+1)$ for all $n \geq N$ (cf. [52]). Moreover, for $m \geq 8$ we have $N \in \{2m+k \mid k \in \{5,6,7\}\}$, as

$$\begin{aligned}\frac{g_m(2m+4)}{g_m(2m+5)} &= \frac{\Gamma(\frac{2m+4}{2})}{\Gamma(\frac{2m+5}{2})}\cdot\frac{\Gamma(\frac{m+5}{2})}{\Gamma(\frac{m+4}{2})}\cdot\frac{\sqrt{2}}{\exp\left(\frac{1}{m}\right)} < 1\ ,\\ \frac{g_m(2m+7)}{g_m(2m+8)} &= \frac{\Gamma(\frac{2m+7}{2})}{\Gamma(\frac{2m+8}{2})}\cdot\frac{\Gamma(\frac{m+8}{2})}{\Gamma(\frac{m+7}{2})}\cdot\frac{\sqrt{2}}{\exp\left(\frac{1}{m}\right)} > 1\ , \qquad \text{for } m \geq 8\ .\end{aligned}$$

(3) For fixed k, the following asymptotics is easily verified:

$$\frac{g_m(2m+k)}{\sqrt{m}} \quad \xrightarrow{m\to\infty} \quad \frac{4\cdot\exp(2)}{\sqrt{2\pi}} < 12\ .$$

In particular, it follows by (2) that for $m \geq 8$ we have an asymptotic estimate of $g_m(n) = O(\sqrt{m})$. More precisely, it is straightforward to check that for $k \in \{5,6,7\}$ and $m \geq 8$ we have $g_m(2m+k) < 2.5 \cdot \sqrt{m}$. It follows by (2) that for $m \geq 8$ we have $g_m(n) < 2.5 \cdot \sqrt{m}$. $\square$

Proof of Theorem 1.4. We will estimate the intrinsic volumes of C via $V_{j-1}(C) \leq v(C) \cdot f_j(n)$, where $f_j(n) = V_{j-1}(\mathcal{L}^n)$ and $v(C)$ denotes the excess over the Lorentz cone introduced in Section 2.3. Note that for $j = m + i - 2k$ with $1 \leq j \leq n-1$, we get from (27) using (21)

$$f_{n-(m+i-2k)}(n) = \frac{\binom{(n-2)/2}{(n-m-i+2k-1)/2}}{2^{n/2}} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-m-i+2k+1}{2})\cdot\Gamma(\frac{m+i-2k+1}{2})\cdot 2^{n/2}}\ . \tag{46}$$

We thus obtain from Corollary 3.3

$$\begin{aligned}
\operatorname{rvol} \mathcal{T}(\Sigma_m,\alpha) &\overset{(38)}{\le} 8 \cdot \sum_{i,k=0}^{n-2} V_{n-m-i+2k-1}(C) \cdot \frac{\Gamma(\frac{m+i-2k+1}{2})}{\Gamma(\frac{m}{2})} \cdot \frac{\Gamma(\frac{n-m-i+2k+1)}{2})}{\Gamma(\frac{n-m}{2})} \\
&\qquad \cdot \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} \cdot I_{n,n-2-i}(\alpha) \\
&\overset{(46)}{\le} 8 \cdot v(C) \cdot \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2}) \cdot \Gamma(\frac{n-m}{2}) \cdot 2^{n/2}} \cdot \sum_{i=0}^{n-2} I_{n,n-2-i}(\alpha) \cdot \sum_{k=0}^{n-2} \binom{m-1}{k} \cdot \binom{n-m-1}{i-k} \\
&= 8 \cdot v(C) \cdot \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2}) \cdot \Gamma(\frac{n-m}{2}) \cdot 2^{n/2}} \cdot \sum_{i=0}^{n-2} I_{n,n-2-i}(\alpha) \cdot \binom{n-2}{i} \\
&\overset{(44)}{<} 8 \cdot v(C) \cdot \frac{\Gamma(\frac{n}{2}) \cdot \exp(\frac{n}{m})}{\Gamma(\frac{m}{2}) \cdot \Gamma(\frac{n-m}{2}) \cdot 2^{n/2}} \cdot \frac{1}{t} , \quad \text{if } t > m \text{ and } n \ge 3 .
\end{aligned}$$

Using the notation from Lemma 3.5, this implies

$$\operatorname{rvol} \mathcal{T}(\Sigma_m,\alpha) < 8 \cdot v(C) \cdot g_m(n) \cdot \frac{1}{t} \overset{(45)}{<} 20 \cdot v(C) \cdot \sqrt{m} \cdot \frac{1}{t} , \quad \text{for } t > m \ge 8 .$$

Analogous to the proof of Theorem 1.3 we estimate the expectation of the logarithm of the Grassmann condition. Defining $\tilde{v}(C) := \max\{v(C), 1\}$, so that in particular $\ln \tilde{v}(C) \ge 0$, we get

$$\begin{aligned}
\mathbb{E}\left[\ln(\mathscr{C}_{\mathrm{G}}(A))\right] &= \int_0^\infty \operatorname{Prob}[\ln(\mathscr{C}_{\mathrm{G}}(A)) > s]\, ds \\
&< \ln(m) + \ln(\tilde{v}(C)) + r + \int_{\ln(m)+\ln(\tilde{v}(C))+r}^\infty 20 \cdot v(C) \cdot \sqrt{m} \cdot \exp(-s)\, ds \\
&= \ln(m) + \ln(\tilde{v}(C)) + r + \frac{20 \cdot v(C)}{\sqrt{m} \cdot \tilde{v}(C) \cdot \exp(r)} \\
&< \ln(m) + \ln(\tilde{v}(C)) + 3 , \quad \text{if } m \ge 8 ,
\end{aligned}$$

where the last inequality follows from choosing $r := \ln\left(\frac{20}{\sqrt{8}}\right)$. □

4 Twisted characteristic polynomials

This section is about a certain subspace-dependent version of the characteristic polynomial of a linear operator. This polynomial will appear in the computation of the volume of the tube around the set Σ_m in the Grassmann manifold. Nevertheless, we will formulate what we call the *twisted characteristic polynomial* in a general context. The proof of the main result of this section is elementary and only involves basic linear algebra.

Let φ be an endomorphism on a k-dimensional euclidean vector space V. We denote by $\sigma_j(\varphi)$, $0 \le j \le k$, the coefficients of the characteristic polynomial of φ (up to sign). More precisely,

$$\det(\varphi - t \cdot \mathrm{id}_V) = \sum_{i=0}^{k} (-1)^{k-i} \cdot \sigma_i(\varphi) \cdot t^{k-i} .$$

Note that we have $\sigma_k(\varphi) = \det(\varphi)$, $\sigma_0(\varphi) = 1$, and $\sigma_1(\varphi) = \mathrm{trace}(\varphi)$. In the following we denote by $\mathrm{Gr}(V, \ell)$ the set of all ℓ-dimensional linear subspaces of V.

Definition 4.1. Let $Y \in \mathrm{Gr}(V, \ell)$, and let Π_Y and $\Pi_{Y^\perp}$ denote the orthogonal projections onto Y and $Y^\perp$, respectively. The *twisted characteristic polynomial* ch_Y and the *positive twisted characteristic polynomial* ch_Y^+ *of* φ *with respect to* Y are defined via

$$\begin{aligned}\mathrm{ch}_Y(\varphi, t) &:= \det\left(\varphi - \left(t \cdot \Pi_Y - \tfrac{1}{t} \cdot \Pi_{Y^\perp}\right)\right) \cdot t^{k-\ell} ,\\ \mathrm{ch}_Y^+(\varphi, t) &:= \det\left(\varphi + \left(t \cdot \Pi_Y + \tfrac{1}{t} \cdot \Pi_{Y^\perp}\right)\right) \cdot t^{k-\ell} .\end{aligned}$$

Furthermore, we denote by $\varphi_Y : Y \to Y$ the *restriction* of φ, i.e., $\varphi_Y(y) := \Pi_Y(\varphi(y))$, and we use the notation $\det_Y(\varphi) := \det(\varphi_Y)$.

Note that for $\ell = k$ we get $\mathrm{ch}_V(\varphi, t) = \det(\varphi - t \cdot \mathrm{id}_V)$, the usual characteristic polynomial, whereas for $\ell = 0$ we get $\mathrm{ch}_0(\varphi, t) = \det(t \cdot \varphi + \mathrm{id}_V)$. We will see below that

$$\mathrm{ch}_Y(\varphi, 0) = \mathrm{ch}_Y^+(\varphi, 0) = \det_Y(\varphi) . \tag{47}$$

Theorem 4.2. *Let V be a k-dimensional euclidean vector space and let φ be an endomorphism on V. If $Y \in \mathrm{Gr}(V, \ell)$ is chosen uniformly at random, then*

$$\mathop{\mathbb{E}}_Y\big[\det_Y(\varphi)\big] = \frac{1}{\binom{k}{\ell}} \cdot \sigma_\ell(\varphi) . \tag{48}$$

Moreover, the expectation of the (positive) twisted characteristic polynomial is given by

$$\mathop{\mathbb{E}}_Y[\mathrm{ch}_Y(\varphi, t)] = \sum_{i,j=0}^{k} d_{ij} \cdot \sigma_{k-j}(\varphi) \cdot t^{k-i} , \quad \mathop{\mathbb{E}}_Y\left[\mathrm{ch}_Y^+(\varphi, t)\right] = \sum_{i,j=0}^{k} |d_{ij}| \cdot \sigma_{k-j}(\varphi) \cdot t^{k-i} , \tag{49}$$

where the coefficients d_{ij} are given for $i + j + \ell \equiv 0 \bmod 2$ and $0 \le \frac{i-j}{2} + \frac{\ell}{2} \le \ell$, $0 \le \frac{i+j}{2} - \frac{\ell}{2} \le k - \ell$, by

$$d_{ij} \;:=\; (-1)^{\frac{i-j}{2} - \frac{\ell}{2}} \cdot \frac{\binom{\ell}{\frac{i-j}{2}+\frac{\ell}{2}} \cdot \binom{k-\ell}{\frac{i+j}{2}-\frac{\ell}{2}}}{\binom{k}{j}} , \tag{50}$$

and $d_{ij} := 0$ otherwise. (Note that $|d_{ij}| = d_{ij}^{k+2,\ell+1}$ in the notation of (36)*.)*

If φ is positive semidefinite, then for $t \ge 0$ we have $|\mathrm{ch}_Y(\varphi, t)| \le \mathrm{ch}_Y^+(\varphi, t)$ and

$$\mathop{\mathbb{E}}_Y\Big[\big|\,\mathrm{ch}_Y(\varphi, t)\big|\Big] \;\le\; \sum_{i,j=0}^{k} |d_{ij}| \cdot \sigma_{k-j}(\varphi) \cdot t^{k-i} .$$

We will now express the twisted characteristic polynomial in coordinates. For $A \in \mathbb{R}^{k \times k}$ and $0 \le \ell \le k$ we use the notation

$$\mathrm{ch}_\ell(A, t) := \mathrm{ch}_{\mathbb{R}^\ell \times 0}(\varphi, t) , \quad \mathrm{ch}_\ell^+(A, t) := \mathrm{ch}_{\mathbb{R}^\ell \times 0}^+(\varphi, t) ,$$

where $\varphi : \mathbb{R}^k \to \mathbb{R}^k$, $x \mapsto Ax$. Note that if A has the block decomposition $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right)$, where $A_1 \in \mathbb{R}^{\ell \times \ell}$, and the other blocks accordingly, then

$$\mathrm{ch}_\ell(A, t) = \det\begin{pmatrix} A_1 - tI_\ell & A_2 \\ tA_3 & tA_4 + I_{k-\ell} \end{pmatrix} , \quad \mathrm{ch}_\ell^+(A, t) = \det\begin{pmatrix} A_1 + tI_\ell & A_2 \\ tA_3 & tA_4 + I_{k-\ell} \end{pmatrix} . \tag{51}$$

From this description we get for $Y = \mathbb{R}^\ell \times 0$ the identity $\mathrm{ch}_Y(\varphi, 0) = \mathrm{ch}^+_Y(\varphi, 0) = \det(A_1) = \det_Y(\varphi)$. This proves (47).

If $Y_0 \in \mathrm{Gr}_{k,\ell}$ is fixed and $Q \in O(k)$ is chosen uniformly at random, then the induced probability distribution on $\mathrm{Gr}_{k,\ell}$ via $Q \mapsto QY_0$ is the uniform distribution. From this it is straightforward to check that if $Q \in O(k)$ and $Y \in \mathrm{Gr}_{k,\ell}$ are chosen uniformly at random, then we have

$$\mathop{\mathbb{E}}_Y [\mathrm{ch}_Y(\varphi, t)] = \mathop{\mathbb{E}}_Q \left[\mathrm{ch}_\ell(Q^T A\, Q, t)\right] \ , \quad \mathop{\mathbb{E}}_Y \left[\mathrm{ch}^+_Y(\varphi, t)\right] = \mathop{\mathbb{E}}_Q \left[\mathrm{ch}^+_\ell(Q^T A\, Q, t)\right] \ . \tag{52}$$

These equalities allow to prove Theorem 4.2 with basic matrix calculus. The main idea is to use the multilinearity of the determinant and the invariance of the coefficients $\sigma_j(A)$ under similarity transformations to show that the coefficients of the expectation of the (positive) twisted characteristic polynomial are linear combinations of the $\sigma_j(A)$. We may then compute the exact linear combinations by choosing particularly nice matrices A, namely scalar multiples of the identity matrix.

In the following we use the notation $[k] := \{1, \ldots, k\}$, and we denote by $\binom{[k]}{i}$ the set of all i-element subsets of $[k]$. For $A \in \mathbb{R}^{k\times k}$ and $J \in \binom{[k]}{i}$, we denote by $\mathrm{pm}_J(A)$ the Jth *principal minor* of A, i.e., $\mathrm{pm}_J(A) := \det(A_J)$, where A_J denotes the submatrix of A which results from selecting the rows and columns of A whose indices lie in J. It is well-known (cf. for example [32, Thm. 1.2.12]) that $\sigma_i(A)$ is the sum of all principal minors of A of size i, i.e.,

$$\sigma_i(A) = \sum_{J\in\binom{[k]}{i}} \mathrm{pm}_J(A) \ . \tag{53}$$

For $0 \le \ell \le k$ the ℓth *leading principal minor* is the principal minor for $J = [\ell]$. We denote it by $\mathrm{lpm}_\ell(A) := \mathrm{pm}_{[\ell]}(A)$. Note that

$$\mathrm{lpm}_\ell(A) = \det_{\mathbb{R}^\ell\times 0}(\varphi) \ .$$

Lemma 4.3. *Let $A \in \mathbb{R}^{k\times k}$, and let $Q \in O(k)$ be chosen uniformly at random. Then, for $J \in \binom{[k]}{\ell}$, we have*

$$\mathop{\mathbb{E}}_Q \left[\mathrm{pm}_J(Q^T \cdot A \cdot Q)\right] \ = \ \mathop{\mathbb{E}}_Q \left[\mathrm{lpm}_\ell(Q^T \cdot A \cdot Q)\right] \ = \ \frac{1}{\binom{k}{\ell}} \cdot \sigma_\ell(A) \ . \tag{54}$$

Proof. For the first equality let $J = \{j_1, \ldots, j_\ell\}$, $j_1 < \ldots < j_\ell$, and let π be any permutation of $[k]$ such that $\pi(i) = j_i$ for all $i = 1, \ldots, \ell$. If M_π denotes the permutation matrix according to π, i.e., $M_\pi \cdot e_i = e_{\pi(i)}$, then $A_J = (M_\pi^T \cdot A \cdot M_\pi)_{[\ell]}$, and therefore $\mathrm{pm}_J(A) = \mathrm{lpm}_\ell(M_\pi^T \cdot A \cdot M_\pi)$. This implies

$$\mathop{\mathbb{E}}_Q \left[\mathrm{pm}_J(Q^T \cdot A \cdot Q)\right] = \mathop{\mathbb{E}}_Q \left[\mathrm{lpm}_\ell(M_\pi^T \cdot Q^T \cdot A \cdot Q \cdot M_\pi)\right] = \mathop{\mathbb{E}}_Q \left[\mathrm{lpm}_\ell(Q^T \cdot A \cdot Q)\right] \ ,$$

where we have used the fact that right multiplication by the fixed element M_π leaves the uniform distribution on $O(k)$ invariant. This also implies

$$\begin{aligned}\mathop{\mathbb{E}}_Q \left[\mathrm{lpm}_\ell(Q^T \cdot A \cdot Q)\right] &= \frac{1}{\binom{k}{\ell}} \cdot \sum_{J\in\binom{[k]}{\ell}} \mathop{\mathbb{E}}_Q \left[\mathrm{pm}_J(Q^T \cdot A \cdot Q)\right] = \frac{1}{\binom{k}{\ell}} \cdot \mathop{\mathbb{E}}_Q \Big[\sum_{J\in\binom{[k]}{\ell}} \mathrm{pm}_J(Q^T \cdot A \cdot Q)\Big] \\ &\overset{(53)}{=} \frac{1}{\binom{k}{\ell}} \cdot \mathop{\mathbb{E}}_Q \left[\sigma_\ell(Q^T \cdot A \cdot Q)\right] = \frac{1}{\binom{k}{\ell}} \cdot \sigma_\ell(A) \ . \end{aligned}$$

□

Proof of Theorem 4.2. The statement (48) follows from (52) and (54). For the equalities in (49) it remains to show that for $Q \in O(k)$ uniformly at random

$$\mathop{\mathbb{E}}_{Q}\left[\mathrm{ch}_\ell(Q^T A Q, t)\right] = \sum_{i,j=0}^{k} d_{ij} \cdot \sigma_{k-j}(A) \cdot t^{k-i} \, , \tag{55}$$

$$\mathop{\mathbb{E}}_{Q}\left[\mathrm{ch}_\ell^+(Q^T A Q, t)\right] = \sum_{i,j=0}^{k} |d_{ij}| \cdot \sigma_{k-j}(A) \cdot t^{k-i} \, . \tag{56}$$

By multilinearity we may write the determinant of a matrix with columns $v_1, \ldots, v_{i-1}, v_i + \alpha \cdot e_i, v_{i+1}, \ldots, v_n$ in the form

$$\det(v_1, \ldots, v_{i-1}, v_i + \alpha \cdot e_i, v_{i+1}, \ldots, v_n) = \det(v_1, \ldots, v_n) + \alpha \cdot \det(v_1, \ldots, v_{i-1}, e_i, v_{i+1}, \ldots, v_n).$$

Using this repeatedly, we generally obtain

$$\det(A + \mathrm{diag}(\alpha_1, \ldots, \alpha_n)) = \sum_{J \subseteq [k]} \mathrm{pm}_J(A) \cdot \prod_{i \in [k] \setminus J} \alpha_i \, .$$

As we have (cf. (51))

$$\begin{aligned} \mathrm{ch}_\ell(A, t) &= t^{k-\ell} \cdot \det(A + \mathrm{diag}(-t, \ldots, -t, 1/t, \ldots, 1/t)) \, , \\ \mathrm{ch}_\ell^+(A, t) &= t^{k-\ell} \cdot \det(A + \mathrm{diag}(t, \ldots, t, 1/t, \ldots, 1/t)) \, , \end{aligned}$$

we may expand the (positive) twisted characteristic polynomial to obtain

$$\mathrm{ch}_\ell(A, t) = \sum_{J \subseteq [k]} (-1)^{c_1(J)} \cdot \mathrm{pm}_J(A) \cdot t^{c_2(J)} \, , \quad \mathrm{ch}_\ell^+(A, t) = \sum_{J \subseteq [k]} \mathrm{pm}_J(A) \cdot t^{c_2(J)} \, , \tag{57}$$

where $c_1, c_2 \colon 2^{[k]} \to \mathbb{N}$ are some integer valued functions on the power set of $[k]$. Averaging the twisted characteristic polynomial thus yields

$$\begin{aligned} \mathop{\mathbb{E}}_{Q}\left[\mathrm{ch}_\ell(Q^T \cdot A \cdot Q, t)\right] &= \sum_{J \subseteq [k]} (-1)^{c_1(J)} \cdot \mathop{\mathbb{E}}_{Q}\left[\mathrm{pm}_J(Q^T \cdot A \cdot Q)\right] \cdot t^{c_2(J)} \\ &\overset{(54)}{=} \sum_{J \subseteq [k]} \frac{(-1)^{c_1(J)}}{\binom{k}{|J|}} \cdot \sigma_{|J|}(A) \cdot t^{c_2(J)} = \sum_{i,j=0}^{k} \tilde{d}_{ij} \cdot \sigma_{k-j}(A) \cdot t^{k-i} \, , \end{aligned}$$

for some rational constants $\tilde{d}_{ij}$. To compute these constants, let us consider the matrices $A = s \cdot I_k$. For this choice of A we have $\mathrm{ch}_\ell(s \cdot I_k, t) = (s-t)^\ell \cdot (1 + s \cdot t)^{k-\ell}$. As $\sigma_{k-j}(s \cdot I_k) = \binom{k}{j} \cdot s^{k-j}$, and $Q^T \cdot s \cdot I_k \cdot Q = s \cdot I_k$, we get

$$(s-t)^\ell \cdot (1 + s \cdot t)^{k-\ell} = \mathrm{ch}_\ell(s \cdot I_k, t) = \mathop{\mathbb{E}}_{Q}\left[\mathrm{ch}_\ell(Q^T \cdot s \cdot I_k \cdot Q, t)\right] = \sum_{i,j=0}^{k} \tilde{d}_{ij} \cdot \binom{k}{j} \cdot s^{k-j} \cdot t^{k-i} \, .$$

We expand the first term in order to make a comparison of the coefficients to get the $\tilde{d}_{ij}$. We have

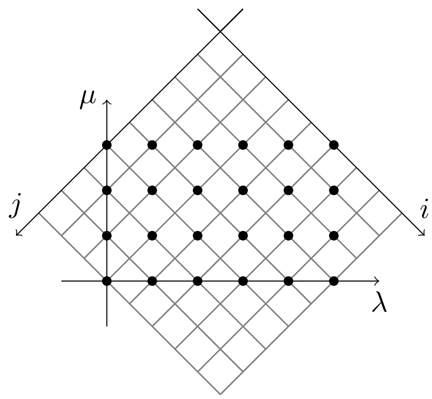


Figure 3: Illustration of the change of summation in (58) ($k = 8$, $\ell = 5$).

$$\begin{aligned}(s-t)^{\ell}\cdot(1+s\cdot t)^{k-\ell} &= \left(\sum_{\lambda=0}^{\ell}\binom{\ell}{\lambda}(-1)^{\ell-\lambda}\cdot s^{\lambda}\cdot t^{\ell-\lambda}\right)\cdot\left(\sum_{\mu=0}^{k-\ell}\binom{k-\ell}{\mu}s^{\mu}\cdot t^{\mu}\right)\\ &= \sum_{\lambda=0}^{\ell}\sum_{\mu=0}^{k-\ell}(-1)^{\ell-\lambda}\binom{\ell}{\lambda}\binom{k-\ell}{\mu}\cdot s^{\lambda+\mu}\cdot t^{\ell-\lambda+\mu} \qquad (58)\\ &\overset{\substack{i=k-\ell+\lambda-\mu\\ j=k-\lambda-\mu}}{=} \sum_{\substack{i,j=0\\ i+j+\ell\equiv 0\\ (\mathrm{mod}\ 2)}}^{k}(-1)^{\frac{i-j}{2}-\frac{\ell}{2}}\binom{\ell}{\frac{i-j}{2}+\frac{\ell}{2}}\binom{k-\ell}{k-\frac{i+j+\ell}{2}}\cdot s^{k-j}\cdot t^{k-i},\end{aligned}$$

where again we interpret $\binom{n}{m} = 0$ if $m < 0$ or $m > n$, i.e., the above summation over i, j in fact only runs over the rectangle determined by the inequalities $0 \le \frac{i-j}{2} + \frac{\ell}{2} \le \ell$ and $0 \le k - \frac{i+j+\ell}{2} \le k - \ell$. See Figure 3 for an illustration of the change of summation. Note that the reverse substitution is given by $\lambda = \frac{i-j+\ell}{2}$ and $\mu = k - \frac{i+j+\ell}{2}$.

Comparing the coefficients of the two expressions of $(s-t)^{\ell}\cdot(1+s\cdot t)^{k-\ell}$ reveals that indeed $\tilde{d}_{ij} = d_{ij}$ as defined in (50). This shows the equality in (55).

The equality in (56) is shown analogously with the observation

$$\mathrm{ch}^{+}_{\ell}(s\cdot I_k, t) = (s+t)^{\ell}\cdot(1+s\cdot t)^{k-\ell}\ .$$

As for the last claim in Theorem 4.2, note that for positive semidefinite A every principal minor is nonnegative, i.e., $\mathrm{pm}_J(A) \ge 0$ for all $J \subseteq [k]$. Therefore, if $t \ge 0$, we get from (57)

$$|\mathrm{ch}_{\ell}(A,t)| \overset{(57)}{=} \Big|\sum_{J\subseteq[k]}(-1)^{c_1(J)}\cdot\mathrm{pm}_J(A)\cdot t^{c_2(J)}\Big| \le \sum_{J\subseteq[k]}\mathrm{pm}_J(A)\cdot t^{c_2(J)} \overset{(57)}{=} \mathrm{ch}^{+}_{\ell}(A,t)\ .$$

This finishes the proof of Theorem 4.2. □

5 Preliminaries from Riemannian geometry

We refer to [19], [8], or [15, Ch. 1], for background on Riemannian geometry. In the following subsections we will focus on three different subjects. First, we will state the smooth coarea

formula, which is the basis for all subsequent volume computations. Second, we will describe the Grassmann manifold as a quotient of the orthogonal group. This viewpoint is a great help in doing explicit calculations in the Grassmann manifold. The third and final topic is a description of the (orthonormal) frame bundle and the Grassmann bundle of a manifold. These bundles form the geometric basis for the computation of the tube formula in Section 6.

5.1 Coarea formula

An important tool in our computations will be the *smooth coarea formula*. Before we can state this, we need to define the normal determinant of a linear operator.

If $A\colon V \to W$ is a linear operator between euclidean vector spaces V and W, then we define the *normal determinant* of A as

$$\operatorname{ndet}(A) := |\det(A|_{\ker(A)^\perp})| \;,$$

where $A|_{\ker(A)^\perp}\colon \ker(A)^\perp \to \operatorname{im}(A)$ denotes the restriction of A to the orthogonal complement of the kernel of A. Obviously, if A is a bijective linear operator then $\operatorname{ndet}(A) = |\det(A)|$, so the normal determinant provides a natural generalization of the absolute value of the determinant.

Theorem 5.1. *Let $\varphi\colon \mathcal{M}_1 \to \mathcal{M}_2$ be a smooth surjective map between Riemannian manifolds $\mathcal{M}_1, \mathcal{M}_2$. Then for any $f\colon \mathcal{M}_1 \to \mathbb{R}$ that is integrable w.r.t. $d\mathcal{M}_1$ we have*

$$\int\limits_{\mathcal{M}_1} f\, d\mathcal{M}_1 = \int\limits_{q\in\mathcal{M}_2} \int\limits_{p\in\varphi^{-1}(q)} \frac{f}{\operatorname{ndet}(D_p\varphi)}\, d\varphi^{-1}(q)\, d\mathcal{M}_2 \;. \tag{59}$$

If additionally $\dim\mathcal{M}_1 = \dim\mathcal{M}_2$, *then*

$$\operatorname{vol}\mathcal{M}_2 := \int\limits_{\mathcal{M}_2} d\mathcal{M}_2 \leq \int\limits_{q\in\mathcal{M}_2} \#\varphi^{-1}(q)\, d\mathcal{M}_2 = \int\limits_{p\in\mathcal{M}_1} \operatorname{ndet}(D_p\varphi)\, d\mathcal{M}_1 \;, \tag{60}$$

where $\#\varphi^{-1}(q)$ denotes the number of elements in the fiber $\varphi^{-1}(q)$. □

The inner integral in (59) over the fiber $\varphi^{-1}(q)$ is well-defined for almost all $q \in \mathcal{M}_2$. This follows from Sard's lemma (cf. for example [50, Thm. 3-14]), which implies that almost all $q \in \mathcal{M}_2$ are regular values, i.e., the differential $D_p\varphi$ has full rank for all $p \in \varphi^{-1}(q)$. The fibers $\varphi^{-1}(q)$ of regular values q are smooth submanifolds of $\mathcal{M}_1$ and therefore the integral over $\varphi^{-1}(q)$ is well-defined. One calls $\operatorname{ndet}(D_p\varphi)$ the *Normal Jacobian* of φ at p.

See [37, 3.8] or [22, 3.2.11] for proofs of the coarea formula with $\mathcal{M}_1, \mathcal{M}_2$ being submanifolds of euclidean space. See [33, Appendix] for a proof of the coarea formula in the above stated form.

5.2 Orthogonal group and Grassmann manifold

The Lie algebra of the orthogonal group is given by

$$T_{I_n}O(n) = \operatorname{Skew}_n := \{U \in \mathbb{R}^{n\times n} \mid U^T = -U\} \;.$$

The tangent space of an element $Q \in O(n)$ is thus given by $T_QO(n) = Q \cdot \operatorname{Skew}_n$.

As for the Riemannian metric on $O(n)$, it is convenient to scale the (euclidean) metric induced from $\mathbb{R}^{n\times n}$ by a factor of $\frac{1}{2}$. The Riemannian metric $\langle .,. \rangle_Q$ on $T_Q O(n) = Q \cdot \mathrm{Skew}_n$ is thus given by

$$\langle QU_1, QU_2 \rangle_Q = \tfrac{1}{2} \cdot \mathrm{tr}(U_1^T \cdot U_2) \ . \tag{61}$$

for $U_1, U_2 \in \mathrm{Skew}_n$. Observe that we have a canonical basis for Skew_n given by $\{E_{ij} - E_{ji} \mid 1 \le j < i \le n\}$, where E_{ij} denotes the (i,j)th elementary matrix, i.e., the matrix whose entries are zero everywhere except for the (i,j)th entry, which is 1. This basis is orthogonal and by the choice of the scaling factor it is also orthonormal.

The *exponential map* of a compact Riemannian manifold $\mathcal{M}$ is a map $\exp\colon T\mathcal{M} \to \mathcal{M}$, where $T\mathcal{M}$ denotes the *tangent bundle* of $\mathcal{M}$, i.e., the disjoint union of all tangent spaces. For $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$, the map $\gamma(t) := \exp_p(tv)$ is the *constant speed geodesic* with $\gamma(0) = p$ in direction $\dot\gamma(0) = v$. We have $d_g(\gamma(t_0), \gamma(t_1)) = |t_0 - t_1| \cdot \|v\|$ for $|t_0 - t_1|$ sufficiently small. Moreover, we have $d_g(\gamma(t_0), \gamma(t_1)) \le |t_0 - t_1| \cdot \|v\|$ for all t_0, t_1.

For the orthogonal group $O(n)$, the exponential map $\exp_Q\colon T_Q O(n) \to O(n)$ at $Q \in O(n)$ is given by the usual matrix exponential, i.e., for $U \in \mathrm{Skew}_n$ we have $\exp_Q(QU) = Q \cdot e^U = Q \cdot \sum_{k=0}^{\infty} \frac{U^k}{k!}$. For example, for

$$N := \left(\begin{array}{c|c|c} 0 & 0 & -1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right) \tag{62}$$

the exponential map at Q in direction QN is given by

$$\exp_Q(\rho \cdot QN) = Q \cdot Q_\rho \ , \quad Q_\rho := \left(\begin{array}{c|c|c} \cos\rho & 0 & -\sin\rho \\ \hline 0 & I_{n-2} & 0 \\ \hline \sin\rho & 0 & \cos\rho \end{array}\right) . \tag{63}$$

Note that $\frac{d}{d\rho} Q_\rho = Q_\rho \cdot N$.

It can be shown (cf. for example [8, §VII.6]) that for sufficiently small $r > 0$ the restriction of the exponential map at Q to an open ball $B_Q(r)$ of radius r around the origin in $T_Q O(n)$, is a diffeomorphism between $B_Q(r)$ and its image. This statement in fact holds for arbitrary Riemannian manifolds $\mathcal{M}$. Furthermore, for arbitrary $r \ge 0$ we have (cf. [8, §VII.6]) for $p \in \mathcal{M}$ and $\bar{B}_p(r) \subset T_p\mathcal{M}$ the closed ball of radius r around the origin in $T_p\mathcal{M}$,

$$\exp_p(\bar{B}_p(r)) = \{q \in \mathcal{M} \mid d_g(p,q) \le r\} \ ,$$

where d_g denotes the geodesic distance in $\mathcal{M}$. Moreover, if $\mathcal{H} \subset \mathcal{M}$ is a compact hypersurface in $\mathcal{M}$ with unit normal vector field ν, then we can characterize the *tube* of radius r around $\mathcal{H}$ as follows:

$$\mathcal{T}(\mathcal{H}, r) := \left\{ q \in \mathcal{M} \mid \min_{p \in \mathcal{H}} d_g(p,q) \le r \right\} = \bigcup_{p \in \mathcal{H}} \exp_p([-r,r] \cdot \nu(p)) \ . \tag{64}$$

(For a proof see for example [51, Addendum to Chap. 9, proof of Thm. 20].)

The *Grassmann manifold* $\mathrm{Gr}_{n,m}$, $1 \le m \le n$, is defined as the set of all m-dimensional subspaces of $\mathbb{R}^n$. We can describe $\mathrm{Gr}_{n,m}$ as the quotient $O(n)/H := \{QH \mid Q \in O(n)\}$ of $O(n)$ by the subgroup

$$H := \left\{ \begin{pmatrix} Q' & 0 \\ 0 & Q'' \end{pmatrix} \middle| \, Q' \in O(m)\,,\; Q'' \in O(n-m) \right\} \simeq O(m) \times O(n-m)$$

in the following way:

$$\begin{tikzcd} O(n) \arrow[r, two heads, "\Pi"] \arrow[d] & \mathrm{Gr}_{n,m} \\ O(n)/H \arrow[ur, dashed, "\sim"'] & \end{tikzcd} \quad , \quad \text{with } \Pi\colon Q \mapsto Q\,(\mathbb{R}^m \times 0)\,. \tag{65}$$

In other words, the identification of $\mathrm{Gr}_{n,m}$ with the quotient $O(n)/H$ amounts to identifying an m-dimensional linear subspace W of $\mathbb{R}^n$ with the set of all orthogonal matrices whose first m columns span W. Note that this identification endows $\mathrm{Gr}_{n,m}$ with a topology, namely the quotient topology on $O(n)/H$.

In the following paragraphs we will give a concrete description of the tangent space $T_W \mathrm{Gr}_{n,m}$ for $W \in \mathrm{Gr}_{n,m}$, and we will describe the Riemannian metric and the exponential map on $\mathrm{Gr}_{n,m}$. Note that the natural action of $O(n)$ on $\mathbb{R}^n$ induces a corresponding action on $\mathrm{Gr}_{n,m}$. It can be shown that, up to scaling, there exists a unique Riemannian metric on $\mathrm{Gr}_{n,m}$, which is $O(n)$-invariant. This is the Riemannian metric, that we will describe next explicitly.

Let $W \in \mathrm{Gr}_{n,m}$ and let $Q \in \Pi^{-1}(W)$, i.e., the first m columns of Q span W. The coset QH is a submanifold of $O(n)$, and its tangent space at Q is given by

$$T_Q QH = Q \cdot T_{I_n} H = Q \cdot \left\{ \begin{pmatrix} U' & 0 \\ 0 & U'' \end{pmatrix} \middle| \, U' \in \mathrm{Skew}_m\,,\; U'' \in \mathrm{Skew}_{n-m} \right\}\,.$$

Note that the orthogonal complement $(T_Q QH)^\perp$ of $T_Q QH$ in $T_Q O(n)$ is given by

$$(T_Q QH)^\perp = Q \cdot \overline{\mathrm{Skew}_n}\,, \text{ where } \overline{\mathrm{Skew}_n} := \left\{ \begin{pmatrix} 0 & -X^T \\ X & 0 \end{pmatrix} \middle| \, X \in \mathbb{R}^{(n-m)\times m} \right\}\,. \tag{66}$$

It can be shown (cf. [30, Lemma II.4.1]) that there exists an open ball B around the origin in $T_Q O(n)$ such that the restriction of $\Pi \circ \exp_Q$ to $B \cap (T_Q QH)^\perp$ is a diffeomorphism onto an open neighborhood of $W = \Pi(Q)$ in $\mathrm{Gr}_{n,m}$. The derivative of $\Pi \circ \exp_Q$ therefore yields a linear isomorphism

$$(T_Q QH)^\perp \xrightarrow{\sim} T_W \mathrm{Gr}_{n,m}\,, \qquad QU \mapsto \xi := D(\Pi \circ \exp_Q)(QU)\,,$$

where $U \in \overline{\mathrm{Skew}_n}$.

The pair (Q,U) with $U \in \overline{\mathrm{Skew}_n}$ thus defines the tangent vector $\xi := D(\Pi \circ \exp_Q(QU))$ in $T_W \mathrm{Gr}_{n,m}$. However, we may also represent the subspace W by a group element Qh, for any $h \in H$. A small computation shows that the pair $(Qh, h^{-1}Uh)$ represents the same tangent vector ξ. More generally, if we define an equivalence relation $\sim$ on $O(n) \times H$ via

$$(Q,U) \sim (Q',U') \;:\Longleftrightarrow\; \exists h \in H\colon (Q',U') = (Qh, h^{-1}Uh)\,,$$

then it follows that the tangent vector $\xi \in T_W \operatorname{Gr}_{n,m}$ can be identified with the equivalence class $[Q,U] := \{(Qh, h^{-1}Uh) \mid h \in H\} \in (O(n) \times H)/\sim$. We have thus obtained the following model for $T_W \operatorname{Gr}_{n,m}$:

$$T_W \operatorname{Gr}_{n,m} \simeq \left\{[Q,U] \mid Q \in \Pi^{-1}(W)\,,\ U \in \overline{\operatorname{Skew}_n}\right\}\ .$$

Note that the derivative $D_Q\Pi$ of Π at Q is given by

$$D_Q\Pi(QU) = [Q, \pi(U)]\ , \tag{67}$$

where $\pi\colon \operatorname{Skew} \to \overline{\operatorname{Skew}_n}$ denotes the orthogonal projection.

We can now define the Riemannian metric on $\operatorname{Gr}_{n,m}$ by setting

$$\big\langle [Q,U_1]\,,\ [Q,U_2]\big\rangle := \langle U_1, U_2\rangle_{I_n} \overset{(61)}{=} \tfrac{1}{2}\cdot \operatorname{tr}(U_1^T \cdot U_2)\ , \tag{68}$$

for $U_1, U_2 \in \overline{\operatorname{Skew}_n}$ (this is clearly a well-defined $O(n)$-invariant Riemannian metric). This way, $\Pi\colon O(n) \to \operatorname{Gr}_{n,m}$ becomes a *Riemannian submersion*, i.e., for every $Q \in O(n)$ the restriction of $D\exp_Q$ to the orthogonal complement of its kernel is an isometry. Using the above description of $D\Pi$ and elementary properties of geodesics (cf. for example [19, Sec. 3.2]), one can show that the exponential map on $\operatorname{Gr}_{n,m}$ is given by

$$\exp_W(\xi) = \Pi(\exp_Q(QU)) = \Pi\big(Q\cdot e^U\big)\ ,\quad \xi = [Q,U] \in T_W \operatorname{Gr}_{n,m}\ . \tag{69}$$

In Definition 1.1 we assigned to a regular cone $C \subset \mathbb{R}^n$ and $1 \le m \le n-1$ the compact subsets $\mathcal{P}_m = \mathcal{P}_m(C)$ and $\mathcal{D}_m = \mathcal{D}_m(C)$ of the Grassmann manifold $\operatorname{Gr}_{n,m}$. In Section 6 we will prove that $\Sigma_m = \mathcal{P}_m \cap \mathcal{D}_m$ is a compact hypersurface in $\operatorname{Gr}_{n,m}$. Furthermore, we will identify a unit normal vector field ν in Σ_m, which points into the component $\mathcal{P}_m$, i.e., for $\alpha > 0$ small enough we have $\exp_W(\alpha\cdot\nu) \in \mathcal{P}_m$ for all $W \in \Sigma_m$. In this situation, we can refine the characterization (64) of the tube around Σ_m to a characterization of the primal tube $\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) = \mathcal{T}(\Sigma_m,\alpha) \cap \mathcal{P}_m$ as follows:

$$\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) = \Big(\bigcup_{W\in\Sigma_m} \exp_W([0,\alpha]\cdot\nu(W))\Big) \cap \mathcal{P}_m\ . \tag{70}$$

The inclusion '$\supseteq$' is clear by (64). As for the other inclusion, let $W' \in \mathcal{P}_m$ with $\alpha := d_g(W', \Sigma_m) > 0$. From (64) we get that a minimum length geodesic between some $W \in \Sigma_m$ and W' has the form $\gamma(\rho) = \exp_W(\delta\cdot\rho\cdot\nu(W))$ with $0 \le \rho \le \alpha$, and $\delta \in \{1,-1\}$. It is sufficient to show that $\delta = 1$. By way of contradiction, assume that $\delta = -1$. Then $W' = \gamma(\alpha) = \exp_W(-\alpha\cdot\nu(W)) \in \mathcal{P}_m$. Since $\gamma(\rho) \in \mathcal{D}_m$ for sufficiently small $0 < \rho < \alpha$, there exists $\rho \le \alpha_0 < \alpha$ such that $\gamma(\alpha_0) \in \mathcal{P}_m \cap \mathcal{D}_m = \Sigma_m$. Therefore, $d_g(W', \Sigma_m) \le d_g(\gamma(\alpha), \gamma(\alpha_0)) \le \alpha - \alpha_0 < \alpha$, which is a contradiction.

We may use the smooth coarea formula (59) to compute the volumes of the orthogonal group and of the Grassmann manifold. Note that the map $\varphi\colon O(n) \to S^{n-1}$, $Q \mapsto Qe_1$, where $e_1 \in \mathbb{R}^n$ denotes the first canonical basis vector, is a Riemannian submersion. In particular, we have $\operatorname{ndet}(D_Q\varphi) = 1$ for all $Q \in O(n)$. Moreover, each fiber of φ is isometric to $O(n-1)$. So an application of the smooth coarea formula (59) shows that $\operatorname{vol} O(n) = \mathcal{O}_{n-1}\cdot \operatorname{vol} O(n-1)$. By induction on n we thus get $\operatorname{vol} O(n) = \prod_{i=0}^{n-1} \mathcal{O}_i$. As for the volume of $\operatorname{Gr}_{n,m}$, recall that by definition of the Riemannian metric on $\operatorname{Gr}_{n,m}$, we have a Riemannian submersion

$\Pi\colon O(n) \to \mathrm{Gr}_{n,m}$ whose fibers are isometric to the direct product $O(m) \times O(n-m)$. The smooth coarea formula (59) thus implies

$$\operatorname{vol}\mathrm{Gr}_{n,m} = \frac{\operatorname{vol} O(n)}{\operatorname{vol} O(m) \cdot \operatorname{vol} O(n-m)} = \frac{\prod_{i=n-m}^{n-1} \mathcal{O}_i}{\prod_{i=0}^{m-1} \mathcal{O}_i} .$$

For later use, we also compute

$$\begin{aligned}\frac{\operatorname{vol}\mathrm{Gr}_{n-2,m-1}}{\operatorname{vol}\mathrm{Gr}_{n,m}} &= \frac{\prod_{i=n-m-1}^{n-3} \mathcal{O}_i}{\prod_{i=0}^{m-2} \mathcal{O}_i} \cdot \frac{\prod_{i=0}^{m-1} \mathcal{O}_i}{\prod_{i=n-m}^{n-1} \mathcal{O}_i} = \frac{\mathcal{O}_{m-1} \cdot \mathcal{O}_{n-m-1}}{\mathcal{O}_{n-2} \cdot \mathcal{O}_{n-1}} \overset{(20)}{=} \frac{m(n-m)}{n} \cdot \frac{\omega_m \cdot \omega_{n-m}}{\mathcal{O}_{n-2} \cdot \omega_n} \\ &\overset{(24)}{=} \frac{m(n-m)}{n} \cdot \binom{n/2}{m/2} \cdot \frac{1}{\mathcal{O}_{n-2}} . \end{aligned} \tag{71}$$

5.3 Frame bundle and Grassmann bundle

In the remainder of this section we will discuss some elementary properties of frame bundles and Grassmann bundles. Note that the only difference between $\mathbb{R}^n$ and a general n-dimensional euclidean space is that the space $\mathbb{R}^n$ has a distinguished orthonormal basis.

Let V be a k-dimensional euclidean vector space. We denote the set of orthonormal bases (frames) in V by $F(V)$, i.e.,

$$F(V) = \{(q_1, \ldots, q_k) \in V^k \mid \forall i, j : \langle q_i, q_j \rangle = \delta_{ij}\} .$$

Note that one can naturally identify $F(\mathbb{R}^n)$ with the orthogonal group $O(n)$. In particular, $F(V)$ has the structure of a smooth (Riemannian) manifold. We recall the ℓth Grassmann manifold $\mathrm{Gr}(V, \ell) = \{Y \subseteq V \mid Y \text{ an } \ell\text{-dimensional linear subspace}\}$, defined for $1 \le \ell \le k$. Again, since $\mathrm{Gr}(V, \ell) \simeq \mathrm{Gr}_{k,\ell}$, the set $\mathrm{Gr}(V, \ell)$ has a well-defined structure of a smooth (Riemannian) manifold. Note that we have the smooth surjective map $F(V) \to \mathrm{Gr}(V, \ell)$, $(q_1, \ldots, q_k) \mapsto \operatorname{lin}\{q_1, \ldots, q_\ell\}$, whose definition is related to (65).

Let M be a k-dimensional smooth Riemannian manifold. The *orthonormal frame bundle* and the ℓth *Grassmann bundle*, $1 \le \ell \le k$, over the manifold M are defined as

$$F(M) = \bigcup_{p \in M} \{p\} \times F(T_pM) , \qquad \mathrm{Gr}(M, \ell) = \bigcup_{p \in M} \{p\} \times \mathrm{Gr}(T_pM, \ell) .$$

Using charts of M to connect the manifolds $F(T_pM)$, it is possible to show that $F(M)$ has the structure of a smooth manifold. Moreover, $F(M) \to M$ is a fiber bundle (cf. for example [35, §I.5]). The same holds true for the Grassmann bundle $\mathrm{Gr}(M, \ell) \to M$. Furthermore, the maps of the fibers $F(T_pM) \to \mathrm{Gr}(T_pM, \ell)$ may be combined to a smooth surjective (bundle) map

$$\Pi_b\colon F(M) \to \mathrm{Gr}(M, \ell) , \quad \Pi_b(p, \bar{Q}) = (p, \operatorname{lin}\{q_1, \ldots, q_\ell\}) , \tag{72}$$

where $\bar{Q} = (q_1, \ldots, q_k)$.

For $(p, \bar{Q}) \in F(M)$ one has a natural decomposition of the tangent space of $F(M)$ at $(p, \bar{Q})$ into a *vertical space* and a *horizontal space*,

$$T_{(p,\bar{Q})}F(M) = T^v_{(p,\bar{Q})}F(M) \oplus T^h_{(p,\bar{Q})}F(M) . \tag{73}$$

The vertical space is defined by

$$T^v_{(p,\bar{Q})}F(M) := T_{(p,\bar{Q})}(\{p\} \times F(T_pM)) \ . \tag{74}$$

For defining the horizontal space we consider the following construction. Let $\zeta \in T_pM$ and let $c\colon \mathbb{R} \to M$ be such that $c(0) = p$ and $\dot{c}(0) = \zeta$. Furthermore, let $\bar{Q}_t$ denote the *parallel transport* (cf. for example [8, Thm. VII.3.12]) of the frame $\bar{Q}$ along c at time t. Then the map $c_F\colon \mathbb{R} \to F(M)$, $t \mapsto (c(t), \bar{Q}_t)$, defines a smooth curve in $F(M)$ and thus a tangent vector $\dot{c_F}(0) \in T_{(p,\bar{Q})}F(M)$. It can be shown (cf. for example [35, §II.3]) that this tangent vector does not depend on the specific choice of the curve c. Moreover, one can show that the induced map $T_pM \to T_{(p,\bar{Q})}F(M)$ sending each tangent vector ζ to the above defined tangent vector of $F(M)$ at $(p,\bar{Q})$, is an injective linear map. The image of this injective linear map is defined as the *horizontal space*. One can then show the decomposition (73) of the tangent space of $F(M)$ at $(p,\bar{Q})$ into the direct sum of the vertical and the horizontal space.

Analogous statements about the decomposition of the tangent spaces also hold for the Grassmann bundle $\mathrm{Gr}(M,\ell)$. Moreover, the derivative of the bundle map $\Pi_b\colon F(M) \to \mathrm{Gr}(M,\ell)$ certainly maps the vertical spaces of $F(M)$ to the vertical spaces of $\mathrm{Gr}(V,\ell)$. As for the horizontal spaces, note that the parallel transport of a linear subspace may be achieved by parallel transporting a basis for this subspace. This implies that also the horizontal spaces of $F(M)$ are mapped to the horizontal spaces of $\mathrm{Gr}(V,\ell)$ via the derivative of the bundle map Π_b.

6 Establishing the tube formula

The following theorem is the main result of this section. Before approaching its proof, we will use it to give a proof of Theorem 3.1.

Theorem 6.1. *Let $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$, $C := \mathrm{cone}(K)$, and let $M := \partial K$.*

1. *The set $\Sigma_m := \Sigma_m(C)$ is an oriented hypersurface of $\mathrm{Gr}_{n,m}$. The map*

$$\Phi_m\colon \mathrm{Gr}(M,m-1) \to \Sigma_m \ , \quad (p,Y) \mapsto \mathbb{R}p + Y \tag{75}$$

   is a diffeomorphism with inverse given by $\Phi_m^{-1}(W) = (p,Y)$, where $W \cap K = \{p\}$ and $Y = p^\perp \cap W$.

2. *For $W \in \Sigma_m$ let $\nu_\Sigma(W) \in T_W\,\mathrm{Gr}_{n,m}$ denote the normal direction pointing inside $\mathcal{D}_m(K)$, and let the maps Π_M and Ψ be defined via*

$$\begin{aligned} \Pi_M\colon \Sigma_m &\to M & \qquad \Psi\colon \Sigma_m \times \mathbb{R} &\to \mathrm{Gr}_{n,m} \\ W &\mapsto p \ , \text{ where } W \cap K = \{p\} \ , & (W,t) &\mapsto \exp_W(\arctan(-t)\cdot \nu_\Sigma(W)) \ . \end{aligned} \tag{76}$$

   Then the Normal Jacobians of Π_M and Ψ are given by

$$\mathrm{ndet}(D_W\Pi_M) = \det{}_Y(\mathcal{W}_p)^{-1} \ , \quad |\det(D_{(W,t)}\Psi)| = (1+t^2)^{-n/2}\cdot \frac{|\mathrm{ch}_Y(\mathcal{W}_p,t)|}{\det_Y(\mathcal{W}_p)} \ , \tag{77}$$

   where $(p,Y) = \Phi_m^{-1}(W)$, and $\mathcal{W}_p$ denotes the Weingarten map of M at p.

Proof of Theorem 3.1. Recall that the intrinsic volumes $V_j(K)$ are continuous in K (cf. Proposition 2.4). Moreover, the set of smooth caps $\mathcal{K}^{\mathrm{sm}}(S^{n-1})$ lies dense in $\mathcal{K}^c(S^{n-1})$; in particular, it lies dense in the set of regular caps $\mathcal{K}^r(S^{n-1})$ (cf. Section 2.1). Furthermore, it is straightforward to check [3, Lemma 6.1.7] that the map

$$\mathcal{K}(S^{n-1}) \to \mathbb{R}\ , \quad K \mapsto \mathrm{rvol}\, \mathcal{T}^{\mathrm{D}}(\Sigma_m(K), \alpha)$$

is continuous and thus, by compactness of $\mathcal{K}(S^{n-1})$, uniformly continuous. Therefore, in order to show the inequality (35), we may assume w.l.o.g. that $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$.

Let the notation be as in Theorem 6.1. As the unit normal field ν_Σ of Σ_m points into the component $\mathcal{D}_m(K)$, i.e., the direction $-\nu_\Sigma$ points into the component $\mathcal{P}_m(K)$, we have for $\tau := \tan\alpha$

$$\begin{aligned}\Psi(\Sigma_m \times [0,\tau]) \supseteq \Psi(\Sigma_m \times [0,\tau]) \cap \mathcal{P}_m(K) &= \Big(\bigcup_{W\in\Sigma_m} \exp_W([0,\alpha]\cdot(-\nu_\Sigma(W)))\Big) \cap \mathcal{P}_m(K) \\ &\overset{(70)}{=} \mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha)\ .\end{aligned}$$

Applying the coarea formula to the map Ψ, and using (77), yields

$$\begin{aligned}\mathrm{vol}\,\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) &\overset{(60)}{\le} \int\limits_{(W,t)\in\Sigma_m\times[0,\tau]} \mathrm{ndet}(D_{(W,t)}\Psi)\, d(\Sigma_m\times\mathbb{R}) \\ &\overset{(77)}{=} \int\limits_{W\in\Sigma_m} \int_0^\tau (1+t^2)^{-n/2}\cdot \frac{|\mathrm{ch}_Y(\mathcal{W}_p,t)|}{\det_Y(\mathcal{W}_p)}\, dt\, dW.\end{aligned}$$

Changing the integration via the coarea formula applied to the map $\Pi_M\colon \Sigma_m \to M$, and using (77), we get

$$\begin{aligned}\mathrm{vol}\,\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) &\le \int\limits_{p\in M}\ \int\limits_{Y\in\mathrm{Gr}(T_pM,m-1)} \int_0^\tau (1+t^2)^{-n/2}\cdot|\mathrm{ch}_Y(\mathcal{W}_p,t)|\, dt\, dY\, dp \\ &= \int\limits_{p\in M}\int_0^\tau \frac{\mathrm{vol}\,\mathrm{Gr}_{n-2,m-1}}{(1+t^2)^{n/2}}\cdot \mathop{\mathbb{E}}_{Y}\Big[\,|\mathrm{ch}_Y(\mathcal{W}_p,t)|\,\Big]\, dt\, dp\ ,\end{aligned}$$

where the expectation is with respect to Y chosen uniformly at random in $\mathrm{Gr}(T_pM, m-1)$. Using Theorem 4.2 with $k=n-2$ and $\ell=m-1$, we obtain

$$\begin{aligned}\mathrm{vol}\,\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) &\le \int\limits_{p\in M}\int_0^\tau \frac{\mathrm{vol}\,\mathrm{Gr}_{n-2,m-1}}{(1+t^2)^{n/2}}\cdot\sum_{i,j=0}^{n-2}\left|d_{ij}^{nm}\right|\cdot\sigma_{n-2-j}(\mathcal{W}_p)\cdot t^{n-2-i}\, dt\, dp \\ &= \mathrm{vol}\,\mathrm{Gr}_{n-2,m-1}\cdot\sum_{i,j=0}^{n-2}\left|d_{ij}^{nm}\right|\cdot\int_0^\tau\frac{t^{n-2-i}}{(1+t^2)^{n/2}}\,dt\cdot\int\limits_{p\in M}\sigma_{n-2-j}(\mathcal{W}_p)\,dp \\ &\overset{(32),(19)}{=} \mathrm{vol}\,\mathrm{Gr}_{n-2,m-1}\cdot\sum_{i,j=0}^{n-2}\left|d_{ij}^{nm}\right|\cdot I_{n,i}(\alpha)\cdot\mathcal{O}_j\cdot\mathcal{O}_{n-2-j}\cdot V_j(K)\ .\end{aligned}$$

So finally we get for rvol $\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) = \frac{\mathrm{vol}\,\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha)}{\mathrm{vol}\,\mathrm{Gr}_{n,m}}$, using the formula (71),

$$\begin{aligned}\mathrm{rvol}\;\mathcal{T}^{\mathrm{P}}(\Sigma_m,\alpha) \le \frac{\mathrm{vol}\,\mathrm{Gr}_{n-2,m-1}}{\mathrm{vol}\,\mathrm{Gr}_{n,m}} \cdot \sum_{i,j=0}^{n-2} \left|d_{ij}^{nm}\right| \cdot I_{n,i}(\alpha)\cdot \mathcal{O}_j \cdot \mathcal{O}_{n-2-j}\cdot V_j(K) \\ \overset{(71)}{=} \frac{2m(n-m)}{n}\cdot \binom{n/2}{m/2} \cdot \sum_{j=0}^{n-2} V_j(K) \cdot \frac{\mathcal{O}_j \cdot \mathcal{O}_{n-2-j}}{2\cdot \mathcal{O}_{n-2}} \cdot \sum_{i=0}^{n-2} \left|d_{ij}^{nm}\right| \cdot I_{n,i}(\alpha) \\ \overset{(24)}{=} \frac{2m(n-m)}{n}\cdot \binom{n/2}{m/2} \cdot \sum_{j=0}^{n-2} V_j(K) \cdot \begin{bmatrix} n-2 \\ j \end{bmatrix} \cdot \sum_{i=0}^{n-2} |d_{ij}^{nm}| \cdot I_{n,i}(\alpha)\;. \qquad \square\end{aligned}$$

The remainder of this section is devoted to the proof of Theorem 6.1. In order to get a clear picture about where the convexity assumption $M = \partial K$ with $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$ comes into play, for the time being we will only assume that $M \subset S^{n-1}$ is a compact orientable hypersurface with a unit normal vector field $\nu\colon M \to S^{n-1}$. (Take M for example to be the smooth boundary of a non-necessarily convex domain of S^{n-1}.)

We shall define a Sigma set $\Sigma_m(M) \subseteq \mathrm{Gr}_{n,m}$, which extends the notion $\Sigma_m(C)$ in the case $M = \partial K$ with $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$ and $C = \mathrm{cone}(K)$. As the general framework of Section 5.2 and Section 5.3 suggests, it is useful to also define a *lifted Sigma set* $\hat{\Sigma}(M) \subseteq O(n)$ such that $\Sigma_m(M) = \Pi(\hat{\Sigma}(M))$, the canonical projection $\Pi\colon O(n) \to \mathrm{Gr}_{n,m}$ being defined as in (65). We first define the lifted Sigma set $\hat{\Sigma}(M)$ before we give the definition of $\Sigma_m(M)$.

In the following we identify the tangent space T_pM for $p \in M$ with the linear subspace $T_pM = p^\perp \cap \nu(p)^\perp$ of $\mathbb{R}^n$, which has codimension 2. If $(p,\bar{Q}) \in F(M)$, then $\bar{Q} = (q_1,\ldots,q_{n-2})$ is an orthonormal basis of T_pM and in the following we shall interpret $\bar{Q}$ as the matrix in $\mathbb{R}^{n\times(n-2)}$ with the columns $q_i \in \mathbb{R}^n$. Note that $(p,\bar{Q},\nu(p)) = (p,q_1,\ldots,q_{n-2},\nu(p)) \in O(n)$. This defines the map

$$\hat{\Phi}\colon F(M) \to O(n)\;, \quad (p,\bar{Q}) \mapsto Q = (p,\bar{Q},\nu(p))\;.$$

We define the lifted Sigma set $\hat{\Sigma}(M)$ as the image of the map $\hat{\Phi}$

$$\hat{\Sigma} := \hat{\Sigma}(M) := \hat{\Phi}(F(M)) = \{Q \in O(n) \mid \exists p \in M : Q\cdot e_1 = p\,,\ Q\cdot e_n = \nu(p)\}\;,$$

where $e_i \in \mathbb{R}^n$ denote the canonical basis vectors. Analogous to the definition of $\hat{\Phi}$, we define for $1 \le m \le n-1$ the map (cf. (75))

$$\Phi_m\colon\ \mathrm{Gr}(M,m-1) \to \mathrm{Gr}_{n,m}\;, \quad (p,Y) \mapsto \mathbb{R}p + Y\;,$$

where $Y \in \mathrm{Gr}(T_pM,m-1)$ is interpreted as an $(m-1)$-dimensional subspace of $\mathbb{R}^n$. Furthermore, we define

$$\Sigma_m := \Sigma_m(M) := \Phi_m(\mathrm{Gr}(M,m-1)) = \{W \in \mathrm{Gr}_{n,m} \mid \exists p \in M : p \in W\,,\ W \subseteq \nu(p)^\perp\}\;.$$

Note that we have $\Pi(\hat{\Sigma}(M)) = \Sigma_m$, where $\Pi\colon O(n) \to \mathrm{Gr}_{n,m}$ denotes the canonical projection (cf. (65)), but we have a strict inclusion $\hat{\Sigma}(M) \subsetneq \Pi^{-1}(\Sigma_m(M))$. We will see below (cf. Corollary 6.3) that $\hat{\Sigma}$ is in fact a smooth submanifold of $O(n)$ without posing any further assumption on M. In Corollary 6.5 we will see that $\Sigma_m(M)$ is a smooth hypersurface in $\mathrm{Gr}_{n,m}$ if $M = \partial K$ for $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$.

The following commutative diagram provides an overview over the relations, which are central for the understanding of Σ_m:

$$\begin{tikzcd} F := F(M) \arrow[r, two heads, "\hat{\Phi}"] \arrow[d, "\Pi_b"] & \hat{\Sigma} \arrow[r, hook] \arrow[d, "\Pi_\Sigma"] & O(n) \arrow[d, "\Pi"] \\ G := \mathrm{Gr}(M, m-1) \arrow[r, two heads, "\Phi_m"] & \Sigma_m \arrow[r, hook] & \mathrm{Gr}_{n,m} \end{tikzcd} \quad , \tag{78}$$

where Π_b is defined as in (72), and Π_Σ is defined as the restriction of Π (cf. (65)) to $\hat{\Sigma}$. Note that if $(p, \bar{Q}) \in F(M)$ with $\bar{Q} = (q_1, \dots, q_{n-2})$, then $\hat{\Phi}(p, \bar{Q}) = Q = (p, q_1, \dots, q_{n-2}, \nu(p)) \in \hat{\Sigma}$ and $\Pi_b(p, \bar{Q}) = (p, Y)$, where Y is the span of $q_1, \dots, q_{m-1}$.

The tangent spaces of the fiber bundles $F(M)$ and $\mathrm{Gr}(M, m-1)$ have natural decompositions into *vertical* and *horizontal components* (cf. Section 5.3)

$$T_{(p,\bar{Q})}F = T^v_{(p,\bar{Q})}F \oplus T^h_{(p,\bar{Q})}F \ , \qquad T_{(p,Y)}G = T^v_{(p,Y)}G \oplus T^h_{(p,Y)}G \ . \tag{79}$$

We denote the images of the vertical and the horizontal space of the frame bundle at $(p, \bar{Q})$ under the derivative $D\hat{\Phi}$ by

$$T^v_Q\hat{\Sigma} := D\hat{\Phi}\left(T^v_{(p,\bar{Q})}F\right) \ , \qquad T^h_Q\hat{\Sigma} := D\hat{\Phi}\left(T^h_{(p,\bar{Q})}F\right) \ . \tag{80}$$

We will give next concrete descriptions of these spaces.

Before we state the next result about the derivative of $\hat{\Phi}$ we define the following linear maps for $Q = (p, \bar{Q}, \nu(p)) \in \hat{\Sigma}$

$$\hat{E}^v \colon \mathrm{Skew}_{n-2} \to \mathrm{Skew}_n \ , \qquad \hat{E}^h \colon \mathbb{R}^{n-2} \to \mathrm{Skew}_n \ , \tag{81}$$

$$\hat{E}^v(\bar{U}) := \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & \bar{U} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \ , \quad \hat{E}^h(a) := \left(\begin{array}{c|c|c} 0 & -a^T & 0 \\ \hline a & 0 & -b \\ \hline 0 & b^T & 0 \end{array}\right) \ , \text{ where } b := \bar{Q}^T \cdot \mathcal{W}_p(\bar{Q}a) \ .$$

Here $\mathcal{W}_p \colon T_pM \to T_pM$ is the Weingarten map of M at p, cf. (8), and we recall that the image of $\bar{Q} \in \mathbb{R}^{n\times(n-2)}$ equals T_pM, as $(p, \bar{Q}, \nu(p)) \in \hat{\Sigma}$. Note that $\hat{E}^v$ is an isometric embedding independent of the manifold M and the element $Q \in \hat{\Sigma}$, while $\hat{E}^h$ is a linear injection, which depends both on M and on Q. For the sake of simplicity, we do not reflect this dependence in the notation. Note further that the images of $\hat{E}^v$ and $\hat{E}^h$ are orthogonal subspaces of Skew_n.

Proposition 6.2. *Let $(p, \bar{Q}) \in F(M)$ and $Q := \hat{\Phi}(p, \bar{Q}) = (p, \bar{Q}, \nu(p)) \in \hat{\Sigma}$. Then we have*

$$T^v_Q\hat{\Sigma} = Q \cdot \mathrm{im}(\hat{E}^v) \ , \qquad T^h_Q\hat{\Sigma} = Q \cdot \mathrm{im}(\hat{E}^h) \ . \tag{82}$$

Proof. Since $p = Qe_1$ and $\nu(p) = Qe_n$, the image of the fiber $\{p\} \times F(T_pM)$ under the map $\hat{\Phi}$ is given by

$$\hat{\Phi}(\{p\} \times F(T_pM)) = \left\{ Q \cdot \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \tilde{Q} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \middle| \tilde{Q} \in O(n-2) \right\} \ .$$

This implies that the image of the vertical space $T^v_{(p,\bar{Q})}F(M)$ under the derivative $D\hat{\Phi}$ is given by $T^v_Q\hat{\Sigma} = Q \cdot \operatorname{im}(\hat{E}^v)$, cf. (74).

As for the horizontal space, let $\xi \in T^h_{(p,\bar{Q})}F(M)$ be represented by the curve $c_F \colon \mathbb{R} \to F(M)$, i.e., $c_F(0) = (p, \bar{Q})$ and $\dot{c_F}(0) = \xi$, which is given in the following way: $c_F(t) = (c(t), \bar{Q}_t)$, where $\bar{Q}_t$ is the parallel transport of $\bar{Q}$ along the curve c (cf. Section 5.3). Let us denote by $\zeta := \dot{c}(0) \in T_pM$ the tangent vector at p defined by c. The image of the curve c_F under $\hat{\Phi}$ is thus given by $\hat{\Phi}(c_F(t)) = (c(t), \bar{Q}_t, \nu(c(t))) =: Q(t)$, and the image of ξ is given by $D\hat{\Phi}(\xi) = \dot{Q}(0)$. It is sufficient to show that $\dot{Q}(0)$ is given by

$$\dot{Q}(0) = Q \cdot \left(\begin{array}{c|c|c} 0 & -a^T & 0 \\ \hline a & 0 & -b \\ \hline 0 & b^T & 0 \end{array}\right) \text{ , where } a = \bar{Q}^T\zeta \text{ ,} \quad b = \bar{Q}^T\mathcal{W}_p(\zeta) \text{ .} \tag{83}$$

To prove (83), note first that as $Q(t) \in O(n)$ and $Q(0) = Q$, we have $\dot{Q}(0) = Q \cdot U$ with $U \in \mathrm{Skew}_n$. Recall that the columns of $\bar{Q}$ form an orthonormal basis of $T_pM = p^\perp \cap \nu(p)^\perp$, hence $Q^T T_pM = e_1^\perp \cap e_n^\perp$. Therefore, $\dot{c}(0) = \zeta = \bar{Q}a$ for some $a \in \mathbb{R}^{n-2}$, which implies $a = \bar{Q}^T\zeta$. The first column of U is thus given by $Ue_1 = Q^T\dot{Q}(0)e_1 = Q^T\zeta = \left(\begin{smallmatrix} 0 \\ a \\ 0 \end{smallmatrix}\right)$. By skew-symmetry of U, this also gives us the first row of U. The zero matrix in the middle follows from the fact that the frame $\bar{Q}$ is parallel transported along c (cf. [8, §VII.3]). Finally, the last column of $\dot{Q}(0)$ is given by $D_p\nu(\zeta)$. This implies that the last column of U is given by

$$Q^T D_p\nu(\zeta) \overset{(8)}{=} -Q^T\mathcal{W}_p(\zeta) \text{ .}$$

As $\mathcal{W}_p(\zeta) \in T_pM = p^\perp \cap \nu(p)^\perp$, the last column of U has the form $\begin{pmatrix} 0 \\ -b \\ 0 \end{pmatrix}$. Therefore $\mathcal{W}_p(\zeta) = Q\left(\begin{smallmatrix} 0 \\ b \\ 0 \end{smallmatrix}\right) = \bar{Q}b$, which implies $b = \bar{Q}^T\mathcal{W}_p(\zeta)$. The last row follows again by skew-symmetry of U. ◻

Corollary 6.3. *The map $\hat{\Phi}$ is an injective immersion of $F(M)$ into $O(n)$. In particular, $\hat{\Sigma}$ is a smooth submanifold of $O(n)$ of codimension $n-1$, and the tangent space $T_Q\hat{\Sigma}$ at $Q \in O(n)$ has the orthogonal decomposition $T_Q\hat{\Sigma} = T^v_Q\hat{\Sigma} \oplus T^h_Q\hat{\Sigma}$.*

Proof. The fact that $\hat{\Phi}$ is a smooth injective map is obvious. Furthermore, by Proposition 6.2, it follows that the derivative $D\hat{\Phi}$ at $(p, \bar{Q})$ has full rank. The map $\hat{\Phi}$ is thus an injective immersion, and as the domain $F(M)$ is compact, it is also an embedding. As for the dimension, we compute

$$\begin{aligned} \dim\hat{\Sigma} &= \dim F(M) = \dim M + \dim O(n-2) = n - 2 + \tfrac{(n-2)(n-3)}{2} = \tfrac{(n-2)(n-1)}{2} \\ &= \dim O(n) - (n-1) \text{ .} \end{aligned}$$

The decomposition of the tangent space into the direct sum $T_Q\hat{\Sigma} = T^v_Q\hat{\Sigma} \oplus T^h_Q\hat{\Sigma}$ follows from the decomposition of the tangent space of the fiber bundle (79). The fact that $T^v_Q\hat{\Sigma}$ and $T^h_Q\hat{\Sigma}$ are orthogonal follows from the description given in Proposition 6.2. ◻

Now that we have a clear description of the lifted Sigma set $\hat{\Sigma}$, we will transfer this description to Σ_m via the projection map Π_Σ (cf. (78)), resp. its derivative $D\Pi_\Sigma$. Analogous to the lifted case we denote the images of the vertical and the horizontal space, cf. (79), of the Grassmann bundle at (p, Y) under the derivative $D\Phi_m$ by

$$T^v_W \Sigma_m := D\Phi_m \left(T^v_{(p,Y)} G \right) , \qquad T^h_W \Sigma_m := D\Phi_m \left(T^h_{(p,Y)} G \right) . \tag{84}$$

Recall that the subspace $\overline{\mathrm{Skew}_n}$ defined in (66) serves as a model for the tangent spaces of $\mathrm{Gr}_{n,m}$. We consider the linear map

$$\alpha \colon \mathbb{R}^{n-2} \to \overline{\mathrm{Skew}_n} , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \left(\begin{array}{c|c} 0 & \begin{array}{c|c} -x_2^T & 0 \\ \hline 0 & -x_1 \end{array} \\ \hline \begin{array}{c|c} x_2 & 0 \\ \hline 0 & x_1^T \end{array} & 0 \end{array} \right) , \tag{85}$$

where $x_1 \in \mathbb{R}^{m-1}$ and $x_2 \in \mathbb{R}^{n-m-1}$, and we denote its image by

$$S_n := \mathrm{im}(\alpha) \subseteq \overline{\mathrm{Skew}_n} .$$

Note that S_n has dimension $n-2$. We now define the following linear maps for $Q = (p, \bar{Q}, \nu(p)) \in \hat{\Sigma}$

$$E^v \colon \mathbb{R}^{(n-m-1)\times(m-1)} \to \overline{\mathrm{Skew}_n} , \qquad E^h \colon \mathbb{R}^{n-2} \to S_n , \tag{86}$$

$$E^v(X) := \left(\begin{array}{c|c} 0 & \begin{array}{c|c} 0 & 0 \\ \hline -X^T & 0 \end{array} \\ \hline \begin{array}{c|c} 0 & X \\ \hline 0 & 0 \end{array} & 0 \end{array} \right) , \qquad E^h(a) := \left(\begin{array}{c|c} 0 & \begin{array}{c|c} -a_2^T & 0 \\ \hline 0 & -b_1 \end{array} \\ \hline \begin{array}{c|c} a_2 & 0 \\ \hline 0 & b_1^T \end{array} & 0 \end{array} \right) ,$$

where again $b := \bar{Q}^T \cdot \mathcal{W}_p(\bar{Q}a)$, and $a = \binom{a_1}{a_2}$, $b = \binom{b_1}{b_2}$ with $a_1, b_1 \in \mathbb{R}^{m-1}$, $a_2, b_2 \in \mathbb{R}^{n-m-1}$. The map E^v is an isometry if the scalar product on $\overline{\mathrm{Skew}_n}$ induced from $\mathbb{R}^{n\times n}$ is scaled by $\frac{1}{2}$. Note that E^v is independent of M and Q. On the other hand, as in the lifted case, the map E^h depends on both M and Q. Unlike in the lifted case, the map E^h need not be injective. We will see below in Proposition 6.4 that it is injective iff the restriction $\mathcal{W}_{p,Y}$ of the Weingarten map $\mathcal{W}_p$ to the subspace $Y = \mathrm{lin}\{q_1, \ldots, q_{m-1}\} \subseteq T_pM$, where $Q = (p, q_1, \ldots, q_{n-2}, \nu(p))$, has full rank. If this is the case, then $\mathrm{im}\, E^h = S_n$.

Note that we have an orthogonal decomposition of $\overline{\mathrm{Skew}_n}$ into

$$\overline{\mathrm{Skew}_n} = \mathrm{im}\, E^v \oplus S_n \oplus \mathbb{R}N , \tag{87}$$

where $N \in \overline{\mathrm{Skew}_n}$ was defined in (62). If we define $\iota \colon \mathbb{R}^{(n-m-1)\times(m-1)} \to \mathrm{Skew}_{n-2}$, via $X \mapsto \left(\begin{smallmatrix} 0 & -X^T \\ X & 0 \end{smallmatrix} \right)$, and if we denote by $\pi \colon \mathrm{Skew}_n \to \overline{\mathrm{Skew}_n}$ the orthogonal projection, then we have the crucial relation

$$E^v(X) = \hat{E}^v(\iota(X)) , \qquad E^h(a) = \pi(\hat{E}^h(a)) . \tag{88}$$

Proposition 6.4. *1. Let $(p, Y) \in \mathrm{Gr}(M, m-1)$ and $W := \Phi_m(p, Y)$, and let $Q \in \hat{\Sigma}$ be such that $\Pi_\Sigma(Q) = W$. Then we have, using the model of tangent spaces of $\mathrm{Gr}_{n,m}$ described in Section 5.2,*

$$T^v_W \Sigma_m = \left\{ [Q, U] \mid U \in \mathrm{im}(E^v) \right\} , \qquad T^h_W \Sigma_m = \left\{ [Q, U] \mid U \in \mathrm{im}(E^h) \right\} . \tag{89}$$

2. The rank of the map E^h is given by

$$\mathrm{rk}(E^h) = n - m - 1 + \mathrm{rk}\,\mathcal{W}_{p,Y} \ ,$$

where $\mathcal{W}_{p,Y}\colon Y \to Y$ is the restriction of the Weingarten map $\mathcal{W}_p$ to the subspace Y of T_pM. Moreover, if $\mathcal{W}_{p,Y}$ has full rank and if we consider S_n to be endowed with the scalar product $\langle U_1, U_2\rangle := \frac{1}{2}\cdot \mathrm{tr}(U_1^T \cdot U_2)$, then

$$|\det(E^h)| = |\det{}_Y(\mathcal{W}_p)| \ , \tag{90}$$

where $\det_Y(\mathcal{W}_p)$ denotes the determinant of $\mathcal{W}_{p,Y}$ (cf. Definition 4.1).

3. The rank of the derivative $D_{(p,Y)}\Phi_m$ is given by

$$\mathrm{rk}\, D_{(p,Y)}\Phi_m = m(n-m) - m + \mathrm{rk}\,\mathcal{W}_{p,Y} \ ,$$

Proof. (1) Since the derivative of the bundle projection Π_b (cf. (72)) maps the vertical spaces of the frame bundle to the vertical spaces of the Grassmann bundle, cf. Section 5.3, we have

$$T_W^v\Sigma_m \overset{(84)}{=} D\Phi_m\left(T_{(p,Y)}^v G\right) \overset{(78)}{=} D\Pi_\Sigma(T_Q^v\hat{\Sigma}) \overset{(82)}{=} D\Pi_\Sigma(Q\cdot \mathrm{im}(\hat{E}^v)) \ .$$

Using $D_Q\Pi_\Sigma(QU) = [Q, \bar{U}]$ (cf. (67)), we may continue as

$$T_W^v\Sigma_m = \left\{[Q,U] \mid U \in \pi(\mathrm{im}(\hat{E}^v))\right\} = \left\{[Q,U] \mid U \in \mathrm{im}(E^v)\right\} \ .$$

Similar arguments show the description (89) of the horizontal space.

(2) By the above choice of scalar product in S_n, the map α defined in (85) is an isometry on its image. Hence, instead of considering E^h we may focus on the map

$$\alpha^{-1}\circ E^h\colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2} \ , \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto \begin{pmatrix} b_1 \\ a_2 \end{pmatrix} \ ,$$

where as usual $b = \bar{Q}^T\cdot \mathcal{W}_p(\bar{Q}a)$, and $a = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)$ with $a_1, b_1 \in \mathbb{R}^{m-1}$, $a_2, b_2 \in \mathbb{R}^{n-m-1}$. Let $\Lambda \in \mathbb{R}^{(n-2)\times(n-2)}$ be the representation matrix of $\mathcal{W}_p$ with respect to the orthonormal basis $q_1,\ldots,q_{n-2}$ of T_pM, i.e., $\Lambda a = \bar{Q}^T\cdot\mathcal{W}_p(\bar{Q}a) = b$. Furthermore, let Λ be decomposed as $\Lambda = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix}$, where $\Lambda_1 \in \mathbb{R}^{(m-1)\times(m-1)}$ and the other blocks accordingly. Then the representation matrix of $\alpha^{-1}\circ E^h$ is given by $\begin{pmatrix} \Lambda_1 & \Lambda_2 \\ 0 & I_{n-m-1} \end{pmatrix}$. As Λ_1 is the representation matrix of $\mathcal{W}_{p,Y}$ with respect to the orthonormal basis $q_1,\ldots,q_{m-1}$ of Y, we obtain

$$\mathrm{rk}(E^h) = \mathrm{rk}(\alpha^{-1}\circ E^h) = n-m-1+\mathrm{rk}(\Lambda_1) = n-m-1+\mathrm{rk}(\mathcal{W}_{p,Y}) \ .$$

Furthermore, if $\mathcal{W}_{p,Y}$ has full rank $\mathrm{rk}\,\mathcal{W}_{p,Y} = m-1$, then

$$|\det(E^h)| = |\det(\alpha^{-1}\circ E^h)| = |\det(\Lambda_1)| = |\det{}_Y(\mathcal{W}_p)| \ .$$

(3) From (1) and (2) we obtain

$$\begin{aligned} \dim T_W^v\Sigma_m &= \mathrm{rk}(E^v) = (m-1)(n-m-1) \ , \\ \dim T_W^h\Sigma_m &= \mathrm{rk}(E^h) = n-m-1+\mathrm{rk}\,\mathcal{W}_{p,Y} \ . \end{aligned}$$

This implies that the rank of the derivative of Φ_m is given by

$$\mathrm{rk}\, D_{(p,Y)}\Phi_m = \dim T_W^v\Sigma_m + \dim T_W^h\Sigma_m = m(n-m-1) + \mathrm{rk}\,\mathcal{W}_{p,Y} \ . \qquad \square$$

Corollary 6.5. *Let $M = \partial K$ with $K \in \mathcal{K}^{\mathrm{sm}}(S^{n-1})$.*

1. *Then Φ_m is an injective immersion of $\mathrm{Gr}(M, m-1)$ into $\mathrm{Gr}_{n,m}$, and Σ_m is a smooth hypersurface of $\mathrm{Gr}_{n,m}$. Furthermore, for $W \in \Sigma_m$, the tangent space of Σ_m at W has the orthogonal decomposition $T_W\Sigma_m = T^v_W\Sigma_m \oplus T^h_W\Sigma_m$.*

2. *If $W = \Pi_\Sigma(Q) \in \Sigma_m$ for $Q \in \hat{\Sigma}$, then the unit normal vector $\nu_\Sigma(W)$ of Σ_m at W, which points into the component $\mathcal{D}_m$, is given by $\nu_\Sigma(W) = [Q, N] \in T_W \mathrm{Gr}_{n,m}$, where $N \in \overline{\mathrm{Skew}_n}$ was defined in (62).*

3. *Let $\Pi_M \colon \Sigma_m \to M$ be given by $W \mapsto p$, where $W \cap K = \{p\}$ (cf. (76)). Then the normal Jacobian of Π_M at $W \in \Sigma_m$ is given by $\mathrm{ndet}(D_W\Pi_M) = \det_Y(\mathcal{W}_p)^{-1}$, where $Y := W \cap p^\perp \in \mathrm{Gr}(T_pM, m-1)$.*

Proof. (1) The injectivity of Φ_m follows from Proposition 2.3. Furthermore, the Weingarten map $\mathcal{W}_p$ of M is positive definite for every $p \in M$. Therefore, every restriction $\mathcal{W}_{p,Y}$ of $\mathcal{W}_p$ to a subspace $Y \in \mathrm{Gr}(T_pM, m-1)$ is positive definite and in particular has full rank. It follows from Proposition 6.4 that Φ_m is an injective immersion. By compactness of the domain $\mathrm{Gr}(M, m-1)$ it follows that Φ_m is an embedding. The image Σ_m is thus a smooth submanifold of dimension $\dim \Sigma_m = \dim \mathrm{Gr}(M, m-1) = m(n-m)-1$. The claimed orthogonal decomposition of $T_W\Sigma_m$ is a consequence of (89) and (87).

(2) It is clear from the orthogonal decompositions $T_W\Sigma_m = T^v_W\Sigma_m \oplus T^h_W\Sigma_m$ and (87) that $\nu_\Sigma(W)$ lies in the orthogonal complement of $T_W\Sigma_m$ in $T_W \mathrm{Gr}_{n,m}$. To see that it points into the component $\mathcal{D}_m$, we consider the exponential map in this direction. We have

$$\exp_W(\rho \cdot \nu_\Sigma(W)) \overset{(69)}{=} \Pi(\exp_Q(QN)) \overset{(63)}{=} \Pi(Q \cdot Q_\rho) \, .$$

The first column of $Q \cdot Q_\rho$ is given by $p_\rho := \cos\rho \cdot p + \sin\rho \cdot \nu(p)$. As the normal vector $\nu(p)$ points inside the cap K, the vector p_ρ lies in the interior of K for $\rho > 0$ small enough. In particular, $\exp_W(\rho \cdot \nu_\Sigma(W)) \in \mathcal{D}_m$ for $\rho > 0$ small enough.

(3) We may lift the function $\Pi_M \colon \Sigma_m \to M$ to the function $\hat{\Pi}_M \colon \hat{\Sigma} \to M$, $Q \mapsto p$, where $Q = (p, q_1, \ldots, q_{n-2}, \nu(p))$. So we have the following commutative diagram

$$\begin{tikzcd} \hat{\Sigma} \arrow[r, two heads, "\Pi_\Sigma"] \arrow[dr, "\hat{\Pi}_M"'] & \Sigma_m \arrow[d, two heads, "\Pi_M"] \\ & M \end{tikzcd} \quad , \qquad \begin{tikzcd} Q \arrow[r, mapsto] \arrow[dr, mapsto] & W \arrow[d, mapsto] \\ & p \end{tikzcd} \quad , \tag{91}$$

where $W = \mathrm{lin}\{p, q_1, \ldots, q_{m-1}\}$. We next show that the orthogonal complement of $\ker D_W\Pi_M$ in $T_W\Sigma_m$ is given by the horizontal space $T^h_W\Sigma_m$.

As the map $\hat{\Pi}_M$ is just the projection onto the first column, the kernel of $D_Q\hat{\Pi}_M$ is given by

$$\ker D_Q\hat{\Pi}_M = \{U \in T_Q\hat{\Sigma} \mid QUe_1 = 0\} \, .$$

It follows that $T^v_Q\hat{\Sigma} \subseteq \ker D_Q\hat{\Pi}_M$ and $T^h_Q\hat{\Sigma} \cap \ker D_Q\hat{\Pi}_M = 0$. Using the commutative diagram (91), we obtain $T^v_W\Sigma_m \subseteq \ker D_W\Pi_M$ and $T^h_W\Sigma_m \cap \ker D_W\Pi_M = 0$. The fact that $T_W\Sigma_m = T^v_W\Sigma_m \oplus T^h_W\Sigma_m$ is an orthogonal decomposition implies that $T^v_W\Sigma_m = \ker D_W\Pi_M$ and hence $T^h_W\Sigma_m = (\ker D_W\Pi_M)^\perp$.

In order to compute $D_W\Pi_M$ on the horizontal space $T^h_W\Sigma_m$, we consider for fixed $Q = (p, q_1, \dots, q_{n-2}, \nu(p)) \in \hat{\Sigma}$ the following diagram

$$\begin{tikzcd} \mathbb{R}^{n-2} \arrow[r, "E^h"] \arrow[d, equal] & S_n \arrow[r, "\beta"] & T^h_W\Sigma_m \arrow[d, "D_W\Pi_M"] \\ \mathbb{R}^{n-2} \arrow[rr, "\gamma"] & & T_pM \end{tikzcd} ,$$

where $\beta(U) = [Q, U]$ for $U \in S_n$, and $\gamma(a) = \sum_{i=1}^{n-2} a_i \cdot q_i$ for $a = (a_1, \dots, a_{n-2})^T \in \mathbb{R}^{n-2}$. Let us check that this diagram is commutative: using

$$D_Q\Pi_\Sigma(Q \cdot \hat{E}^h(a)) \overset{(67)}{=} [Q, \pi(\hat{E}^h(a))] \overset{(88)}{=} [Q, E^h(a)] ,$$

we get

$$D_W\Pi_M([Q, E^h(a)]) = D_W\Pi_M(D_Q\Pi_\Sigma(Q \cdot \hat{E}^h(a))) \overset{(91)}{=} D_Q\hat{\Pi}_M(Q \cdot \hat{E}^h(a)) = \gamma(a) .$$

Since β and γ are isometric we obtain

$$\mathrm{ndet}(D_W\Pi_M) = |\det(E^h)|^{-1} \overset{(90)}{=} |\det_Y(\mathcal{W}_p)|^{-1} = \det_Y(\mathcal{W}_p)^{-1} ,$$

where the last equality follows from the positive definiteness of $\mathcal{W}_p$. □

For the proof of Theorem 6.1 we make yet another definition in the lifted setting. For $Q \in \hat{\Sigma}$ we define the direction $\hat{\nu}(Q) \in T_QO(n)$ and the map $\hat{\Psi}\colon \hat{\Sigma} \times \mathbb{R} \to O(n)$ via

$$\hat{\nu}(Q) := Q \cdot N , \qquad \hat{\Psi}(Q, t) = \exp_Q(\arctan(-t) \cdot \hat{\nu}(Q)) ,$$

where $N \in \mathrm{Skew}_n$ is, as usual, defined in (62). Abbreviating $\rho := \arctan(-t)$, we obtain from (63) that the map $\hat{\Psi}$ is given by $\hat{\Psi}(Q, t) = Q \cdot Q_\rho$, where Q_ρ was defined in (63). Furthermore, denoting $\tilde{\Pi} := \Pi_\Sigma \times \mathrm{id}_\mathbb{R}$, we get the following commutative diagram

$$\begin{tikzcd} \hat{\Sigma} \times \mathbb{R} \arrow[r, "\hat{\Psi}"] \arrow[d, "\tilde{\Pi}"] & O(n) \arrow[d, "\Pi"] \\ \Sigma_m \times \mathbb{R} \arrow[r, two heads, "\Psi"] & \mathrm{Gr}_{n,m} \end{tikzcd} . \tag{92}$$

Proof of Theorem 6.1. The first part of the theorem, as well as the claim about the normal Jacobian of Π_M, is covered by Corollary 6.5.

It remains to show the claim about the normal Jacobian of Ψ. We will compute the derivative $D\Psi$ via the lifting $D\hat{\Psi}$ in the following way. Let $W \in \Sigma_m$ and let $Q = (p, \bar{Q}, \nu(p)) \in \hat{\Sigma}$ be a lifting of W, i.e., $W = \Pi_\Sigma(Q)$. For $(\xi, \dot{t}) \in T_{(W,t)}(\Sigma_m \times \mathbb{R})$ let $\hat{\xi} \in T_Q\hat{\Sigma}$ be a lifting of ξ, i.e., $\xi = D\Pi_\Sigma(\hat{\xi})$. Using the relations displayed in the commutative diagram (92), we get

$$D\Psi(\xi, \dot{t}) = D\Psi(D\tilde{\Pi}(\hat{\xi}, \dot{t})) \overset{(92)}{=} D\Pi(D\hat{\Psi}(\hat{\xi}, \dot{t})) . \tag{93}$$

From the explicit form $\hat{\Psi}(Q,t) = Q \cdot Q_\rho$, where $\rho = \arctan(-t)$ and Q_ρ defined as in (63), and $\frac{d\rho}{dt} = -(1+t^2)^{-1}$, and from $\frac{d}{d\rho} Q_\rho = Q_\rho N$, we get

$$D_{(Q,t)}\hat{\Psi}(0,1) = -(1+t^2)^{-1} \cdot Q \cdot Q_\rho \cdot N \,.$$

Using (93), we thus have

$$\begin{aligned} D_{(W,t)}\Psi(0,1) &= D\Pi(-(1+t^2)^{-1} \cdot Q \cdot Q_\rho \cdot N) \\ &\overset{(67)}{=} -(1+t^2)^{-1} \cdot [Q \cdot Q_\rho, N] \,. \end{aligned} \tag{94}$$

It remains to compute the derivative of Ψ in the first component.

Taking into account the orthogonal decomposition $T_W\Sigma_m = T_W^v\Sigma_m \oplus T_W^h\Sigma_m$, cf. Corollary 6.5, we first consider the vertical space $T_W^v\Sigma_m$. By (89) every element $\xi \in T_W^v\Sigma_m$ is of the form $\xi = [Q, E^v(X)]$ for some $X \in \mathbb{R}^{(n-m-1)\times(m-1)}$. Furthermore, a lifting $\hat{\xi}$ of ξ is given by $\hat{\xi} = Q \cdot E^v(X)$, as $D\Pi_\Sigma([Q, E^v(X)]) = Q \cdot E^v(X)$, cf. (67). As for fixed ρ, the map $Q \mapsto Q \cdot Q_\rho$ is linear, we obtain from $\hat{\Psi}(Q,t) = Q \cdot Q_\rho$

$$D_{(Q,t)}\hat{\Psi}\big(\hat{\xi}, 0\big) = Q \cdot E^v(X) \cdot Q_\rho \overset{(*)}{=} Q \cdot Q_\rho \cdot E^v(X) \,,$$

where the equality $(*)$ follows from the fact that Q_ρ only acts on the first and the last columns or rows. This implies via (93) and (67)

$$D_{(W,t)}\Psi(\xi, 0) = [Q \cdot Q_\rho \,,\; \pi(E^v(X))] = [Q \cdot Q_\rho \,,\; E^v(X)] \,, \tag{95}$$

where, as usual, $\pi\colon \mathrm{Skew}_n \to \overline{\mathrm{Skew}_n}$ denotes the orthogonal projection.

As for the horizontal space, any element $\zeta \in T_W^v\Sigma_m$ is of the form $\zeta = [Q, E^h(a)]$ for some $a \in \mathbb{R}^{n-2}$, cf. (89). A lifting $\hat{\zeta}$ of ζ is given by $\zeta = Q \cdot \hat{E}^h(a)$, as

$$D\Pi_\Sigma(\hat{\zeta}) \overset{(67)}{=} [Q, \pi(\hat{E}^h(a))] \overset{(88)}{=} [Q, E^h(a)] = \zeta \,.$$

We thus get from $\hat{\Psi}(Q,t) = Q \cdot Q_\rho$

$$D_{(Q,t)}\hat{\Psi}\big(\hat{\zeta}, 0\big) = Q \cdot \hat{E}^h(a) \cdot Q_\rho = Q \cdot Q_\rho \cdot Q_\rho^T \cdot \hat{E}^h(a) \cdot Q_\rho \,.$$

This implies via (93) and (67)

$$D_{(W,t)}\Psi(\zeta, 0) = \Big[Q \cdot Q_\rho \,,\; \pi\left(Q_\rho^T \cdot \hat{E}^h(a) \cdot Q_\rho\right)\Big] \,. \tag{96}$$

To finish the computation, note that we have for $a, b \in \mathbb{R}^{n-2}$

$$Q_\rho^T \cdot \left(\begin{array}{c|c|c} 0 & -a^T & 0 \\ \hline a & 0 & -b \\ \hline 0 & b^T & 0 \end{array}\right) \cdot Q_\rho = \left(\begin{array}{c|c|c} 0 & -ca^T + sb^T & 0 \\ \hline ca - sb & 0 & -sa - cb \\ \hline 0 & sa^T + cb^T & 0 \end{array}\right) \,, \tag{97}$$

where we use the abbreviations $s := \sin(\rho)$, $c := \cos(\rho)$. Using the decompositions $a = \binom{a_1}{a_2}$, $b = \binom{b_1}{b_2}$ with $a_1, b_1 \in \mathbb{R}^{m-1}$, $a_2, b_2 \in \mathbb{R}^{n-m-1}$, we obtain

$$
\begin{aligned}
&\pi\left(Q_\rho^T \left(\begin{array}{c|c|c} 0 & -a^T & 0 \\ \hline a & 0 & -b \\ \hline 0 & b^T & 0 \end{array}\right) Q_\rho\right) \\
&\qquad\overset{(97)}{=} \left(\begin{array}{c|c} 0 & \begin{array}{c|c} -ca_2^T + sb_2^T & 0 \\ \hline 0 & -sa_1 - cb_1 \end{array} \\ \hline \begin{array}{c|c} ca_2 - sb_2 & 0 \\ \hline 0 & sa_1^T + cb_1^T \end{array} & 0 \end{array}\right) .
\end{aligned} \tag{98}
$$

Combining (98) with the formula in (96), we get an explicit formula for the derivative of Ψ on the horizontal space.

To summarize the results from (94), (95), and (96) let us define $W_\rho := \Pi(Q \cdot Q_\rho)$, and set

$$
T^1_{W_\rho} := D\Psi(0 \times \mathbb{R}) , \qquad T^2_{W_\rho} := D\Psi(T^v_W \Sigma_m \times 0) , \qquad T^3_{W_\rho} := D\Psi(T^h_W \Sigma_m \times 0) .
$$

Recall the orthogonal decomposition $\overline{\mathrm{Skew}_n} = \mathbb{R}N \oplus \operatorname{im} E^v \oplus \operatorname{im} E^h$, cf. (87). This implies the orthogonal decomposition

$$
T_{W_\rho} \mathrm{Gr}_{n,m} = T^1_{W_\rho} \oplus T^2_{W_\rho} \oplus T^3_{W_\rho} .
$$

Furthermore, the corresponding restrictions of $D\Psi$ yield a (reflection and a) dilation by $(1+t^2)^{-1}$ between $0 \times \mathbb{R}$ and $T^1_{W_\rho}$ (cf. (94)), an isometry between $T^v_W \Sigma_m \times 0$ and $T^2_{W_\rho}$ (cf. (95)), and a nontrivial linear map $\overline{D\Psi}$ between $T^h_W \Sigma_m \times 0$ and $T^3_{W_\rho}$ (cf. (96), (98)). This implies

$$
\operatorname{ndet}(D_{(W,t)}\Psi) = (1+t^2)^{-1} \cdot |\det \overline{D\Psi}| . \tag{99}
$$

We determine now $|\det \overline{D\Psi}|$.

We consider the following commutative diagram defining the linear map $\mu\colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$:

$$
\begin{tikzcd}
T^h_W \Sigma_m \times 0 \arrow[r, "\overline{D\Psi}"] & T^3_{W_\rho} \\
\mathbb{R}^{n-2} \arrow[u, "\beta \circ E^h"] \arrow[r, dashed, "\mu"] & \mathbb{R}^{(m-1)+(n-m-1)} \arrow[u, "\beta \circ \alpha"']
\end{tikzcd} ,
$$

where $\beta\colon \mathrm{Skew}_n \to T^h_W \Sigma_m$, $U \mapsto [Q, U]$, and $\alpha\colon \mathbb{R}^{n-2} \to \mathrm{Skew}_n$ as defined in (85). As α, β are isometries, we obtain

$$
|\det \overline{D\Psi}| = |\det(\beta \circ E^h)^{-1}| \cdot |\det \mu| = |\det E^h|^{-1} \cdot |\det \mu| \overset{(90)}{=} \det_Y(\mathcal{W}_p)^{-1} \cdot |\det \mu| . \tag{100}
$$

As for the linear map μ, we obtain from (98) that μ has the transformation matrix $\begin{pmatrix} c\Lambda_1 + sI_{m-1} & c\Lambda_2 \\ -s\Lambda_3 & cI_{n-m-1} - s\Lambda_4 \end{pmatrix}$, as $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and

$$
\begin{pmatrix} c\Lambda_1 + sI_{m-1} & c\Lambda_2 \\ -s\Lambda_3 & cI_{n-m-1} - s\Lambda_4 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} sa_1 + cb_1 \\ ca_2 - sb_2 \end{pmatrix} .
$$

Using that $\frac{s}{c} = \tan\rho = -t$ and $c = \cos\rho = 1/\sqrt{1+t^2}$, we obtain

$$\begin{aligned}\det\mu &= \det\begin{pmatrix} c\Lambda_1 + sI_{m-1} & c\Lambda_2 \\ -s\Lambda_3 & cI_{n-m-1} - s\Lambda_4 \end{pmatrix} \\ &= c^{n-2}\cdot\det\begin{pmatrix} \Lambda_1 - tI_{m-1} & \Lambda_2 \\ t\Lambda_3 & I_{n-m-1} + t\Lambda_4 \end{pmatrix} \\ &= (1+t^2)^{-(n-2)/2}\cdot \mathrm{ch}_Y(\mathcal{W}_p, t)\ ,\end{aligned}$$

where the last equality is a consequence of (51). From (99) and (100) we thus get

$$\mathrm{ndet}(D_{(W,t)}\Psi) = (1+t^2)^{-1}\cdot\frac{|\det\mu|}{\det_Y \mathcal{W}_p} = (1+t^2)^{-n/2}\cdot\frac{|\,\mathrm{ch}_Y(\mathcal{W}_p,t)|}{\det_Y\mathcal{W}_p}\ ,$$

which finishes the proof of Theorem 6.1. □

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